DISSOLUTION OF TRANSFORMATION PROBLEM IN THREE-DEPARTMENTS MODEL OF SIMPLE PRODUCTION[1].

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ABSTRACT.

The dissolution of transformation problem in three-departments model of simple production is proposed. Four variants of this model (models-1,2,3,4) are most realistic. Models-1 and 2 don't take into account the "labor of capitalists" (as managers and entrepreneurs). Models-3 and 4 take into account this factor. Embodied labor (value) in models-1 and 2 consists of labor of workers. Embodied labor (value) in models-3 and 4 consists of labor of both workers and capitalists[2]. Capitalists are expending net profit (= gross profit - salary of capitalists as entrepreneurs) on "luxury goods" in models-1 and 3 and on "luxury goods" and "subsistence goods" in models-2 and 4. The model-4 is the most realistic model of capitalist simple production. The solution of transformation problem exists in the models-1 and 2 if non-trivial balance-conditions are superimposed onto the economy. Marx has introduced non-trivial balance-conditions for the simple production in XX chapter of second volume of "Capital". We argue that these conditions were carried out in early capitalist economy (merchant capitalism)[3]. The problem statement in Bortkiewicz' (1907) paper (in frame of model-1) doesn't take into account Marx's non-trivial balance-conditions. The process of historical transformation is considered theoretically and it is modeled numerically.

[1] I thank my daughter Vera Pushnaya for the help in correction of translation of this article. I thank also the participants in dispute devoted to "transformation problem" on scientific forum "Socintegrum" (Valeriy Kaluzhniy and Viktor Shuliga ("Ingener")).
[2] We don't consider in this paper the models-3 and 4. The model-4 is considered in paper Pushnoi (2011).
[3] The discussion of problem why "non-trivial balance-conditions" were carried out in early capitalist economy is in paper Pushnoi (2011).

I. Introduction.

Bortkiewicz (1907) in 1907 has considered the transformation of “values” into “prices of production” in three-departments model of simple production (“means of production” (I); “consumer goods” (II) and “luxury goods” (III)). He found that Marx’s transformational rules couldn’t be executed simultaneously in the economy with arbitrary structure. Only one rule of transformation can be executed in the general case: either the sum of profits is equals to aggregate surplus value or price and value of aggregate output are equal. Bortkiewicz concluded about logical inconsistence in Marx’s theory. So there was the “transformation problem”.

This discouraging result shattered confidence in Marx’s “labor theory of value”. Later many scientists came to the same conclusion (Sweezy (1949); Meek (1956); Samuelson (1957, 1971); Medio (1972); Steedman, I. (1977); Abraham-Frois, G. (1979); Itoh (1980)) and now after a century majority of economists is convinced that Marx's theory is internally inconsistent.

Numerous attempts have been undertaken in order to dissolve this problem. Morishima and Caterhores (1975) have assumed that Marxian algorithm of transformation is only the first step of iterative process. Shaikh (1977; 1984) has assumed that the solution of this problem can be obtained as a result of many iterations of the Marx’s transformation algorithm. Sweezy (1949) has assumed that the problem can be dissolved if in reality “values” and “prices of production” are connected by nonlinear relation. Lipietz (1982), Dumenil (1980; 1983), and Foley (1982) have offered new interpretation of “transformational rules” – so-called “new solution”. Freeman (1996), Kliman and McGlone (1999) and Kliman (2007) have offered so-called "temporal single-system interpretation" (TSSI) in which "output prices" and "input prices" can differ in each period because the real economy is non-equilibrium dynamical system.

Despite elegance of many ideas offered for dissolution of this problem, among Marxists-theorists till now there is not any consent concerning the given problem. Moseley (1999), Fine et al. (2004), Mariolis (2006) and a number of other authors criticize "new solution", noticing that it differs from initial statement of this problem and comprises the implicit tautology. Laibman (2004), Roberto (2004), Park (2009) recently have put forward serious objections against TSSI-approach to a solution of transformation problem. So, the transformation problem still remains the problem which does not have any conventional solution.

We reconsider once again Bortkiewicz’ model. Was the problem stated correctly? The model of simple reproduction for the early capitalist economy assumes the performance of certain non-trivial balance-conditions in this economy. We argue in this paper that Bortkiewicz’ problem statement doesn’t take into account these non-trivial conditions of exchange between departments. As consequence the set of solutions in Bortkiewicz’ paper is wider than it was possible for early capitalist economy with simple reproduction. Superfluous solutions obtained by Bortkiewicz don't satisfy to all Marx’s rules of transformation. Marxian transformational rules are carried out if non-trivial balance-conditions are taken into account.

Paper is structured as follows. Section II contains general description of three-departments models of simple production. Section III is devoted to logical analysis of “value-composition” of economy. We introduce “value-matrix” and “matrix of input-flows” for different types of exchange. Sections IV-V are devoted to consideration of three partial cases: 1) exchange on the base of “values”; 2) exchange on the base of “prices of production”; and 3) exchange at which “values” and “prices of production” coincide. Each mode of exchange corresponds to the definite “value-structure”. We prove that Marxian transformational rules are carried out in the economy with non-trivial balance-conditions if equilibrium prices are equal to "prices of production". We discuss in Section VI why early capitalist economy with simple production must satisfy to NON-TRIVIAL conditions of balanced exchange between departments. We demonstrate that these non-trivial balance-conditions follow from Marx's analysis of simple reproduction in the second volume of "Capital". Section VII is devoted to analysis of Bortkiewicz' (1907a) problem statement. Section VIII discusses the possible variants of "historical transformation". We consider two modes of "historical transformation" and construct the numerical model for each case. Section IX contains the dissolution of transformation problem in model-2 in which capitalists consume both "consumer goods" and "luxury goods". Supplements to this paper contain the numerical examples of the different "value-structures", results of modeling, and Excel-file with program of calculations.

II. Three-departments Model of Simple Production.

Model-1. Capitalists buy and consume only "luxury goods". Profits don't contain any compensation of capitalists as employees (managers and directors at their own enterprises). Workers buy and consume "consumer goods".

Model-2. Capitalists buy and consume both "luxury goods" and "consumer goods". Profits don't contain any compensation of capitalists as employees (managers and directors at their own enterprises). Workers buy and consume only "consumer goods".

Model-3. Capitalists buy and consume only "luxury goods". Part of profit is the compensation ("wage") of capitalists as employees (managers and directors at their own enterprises). Workers buy and consume "consumer goods".

Model-4. Capitalists buy and consume both "luxury goods" and "consumer goods". Part of profit is the compensation ("wage") of capitalists as employees (managers and directors at their own enterprises). Workers buy and consume only "consumer goods".

Bortkiewicz (1907) considered model-1. Marx in the second volume of "Capital" (chapter XX) has considered the Model-2. We consider in this paper the model-1, model-2 and model-4.

Model-4 is the most realistic model of capitalist economy. Capitalists are proprietors but they are also managers and directors at their own enterprises. They work as well as employees at their enterprises. Profit contains "wage" of capitalists. Marx emphasized many times this point in third and fourth volumes of "Capital".

"…the process of production, separated from capital, is simply a labour-process. Therefore, the industrial capitalist, as distinct from the owner of capital, does not appear as operating capital, but rather as a functionary irrespective of capital, or, as a simple agent of the labour-process in general, as a labourer, and indeed as a wage-labourer…

…The specific functions which the capitalist as such has to perform, and which fall to him as distinct from and opposed to the labourer, are presented as mere functions of labour. He creates surplus-value not because he works as a capitalist, but because he also *works, regardless of his capacity of capitalist. This portion of surplus-value is thus no longer surplus-value, but its opposite, an equivalent for labour performed…*

The conception of profit of enterprise as the wages of supervising labour, arising from the antithesis of profit of enterprise to interest, is further strengthened by the fact that a portion of profit may, indeed, be separated, and is separated in reality, as wages, or rather the reverse, that a portion of wages appears under capitalist production as integral part of profit…" (K. Marx "Capital", v. III, ch. XXIII)

"…[Consequently] the industrial capitalist as distinct from himself as capitalist, that is, the industrialist in contradistinction to himself as capitalist, i.e., owner of capital, is thus merely a simple functionary in the labour process; he does not represent functioning capital, but is a functionary irrespective of capital, and therefore a particular representative of the labour process in general, a worker. In this way,

industrial profit is happily converted into wages *and is equated with ordinary wages, differing from them only quantitatively and in the special form in which they are paid, i.e., that the capitalist pays wages to himself instead of someone else paying them to him…*

…Therefore, insofar as the capitalist plays any part in it, he does so not as a capitalist—for this aspect of his character is allowed for in interest—but as a functionary of the labour process in general, as a worker, and his wages take the form of industrial profit. It is a special type of labour—labour—of superintendence—but after all types of labour in general differ from one another.

Industrial profit does indeed include some part of wages—in those cases where the manager does not draw them. Capital appears in the production process as the director of labour, as its commander (captain of industry) and thus plays an active role in the labour process… This work (it may be entrusted to a manager) which is linked with exploitation is, of course, labour which, in the same way as that of the wage-worker, enters into the value of the product…" (K. Marx, "Capital", v. IV ("Theories of surplus-values"), part. III, Addenda 4).

Capitalists in real life buy both "consumer goods" and "luxury goods". Moreover the profit of capitalists consists of "net profit" and "wage" of capitalists as entrepreneurs ("profit" = "wage of capitalists" + "net profit"). Consequently models-1 and 3 are very simplified (crude) models. The model-4 is the most realistic model of real capitalist economy in frame of three-departments model.

Let's introduce two matrices: 1) matrix of value-structure and 2) matrix of input-flows. The first matrix describes values of commodities which are consumed by each department during the production period. Workers in each department buy "consumer goods", capitalists buy "means of production" (equipment, raw materials, fuel, semi-finished products) "consumer goods" and "luxury goods". The second matrix contains these articles of expenses in prices of balanced exchange. Multipliers $x;y;z$ connect "values" and "prices". "Value" of output is so-called "labor cost" of output (the labor expended during the production process and embodied in commodities). "Value" of department' output is not equal in general case to the sum of "values" of commodities consumed in department during the production process (so-called "labor commanded").

$$\underbrace{C_1 + C_2 + C_3 = C}_{\text{"labor cost"}} \neq \underbrace{C_1 + V_1 + M_1}_{\text{"labor commanded"}} \qquad (1)$$

$$\underbrace{V_1 + V_2 + V_3 = V}_{\text{"labor cost"}} \neq \underbrace{C_2 + V_2 + M_2}_{\text{"labor commanded"}} \qquad (2)$$

$$\underbrace{M_1 + M_2 + M_3 = M}_{\text{"labor cost"}} \neq \underbrace{C_3 + V_3 + M_3}_{\text{"labor commanded"}} \qquad (3)$$

Table 1. Matrix I of value-structure.

Dept.	Value of "means of production". (1)	Value of "consumer goods". (2)	Value of "luxury goods". (3)	Sum of values consumed in department ("labor commanded"). (4)	Value of output ("labor cost"). (5)
I.	C_1	V_1	M_1	$C_1+V_1+M_1$	C
II.	C_2	V_2	M_2	$C_2+V_2+M_2$	V
III.	C_3	V_3	M_3	$C_3+V_3+M_3$	M
Σ	$C=\sum_{n=1}^{3} C_n$	$V=\sum_{n=1}^{3} V_n$	$M=\sum_{n=1}^{3} M_n$	$C+V+M$	$C+V+M$

Table 2. Matrix II of input-flows.

Dept.	Price of "means of production". (1)	Price of "consumer goods". (2)	Price of "luxury goods". (3)	Sum of prices of goods consumed in department. (4)	Price of output. (5)
I.	xC_1	yV_1	zM_1	$xC_1+yV_1+zM_1$	xC
II.	xC_2	yV_2	zM_2	$xC_2+yV_2+zM_2$	yV
III.	xC_3	yV_3	zM_3	$xC_3+yV_3+zM_3$	zM
Σ	$xC=x\sum_{n=1}^{3} C_n$	$yV=y\sum_{n=1}^{3} V_n$	$zM=z\sum_{n=1}^{3} M_n$	$xC+yV+zM$	$xC+yV+zM$

These relations in price-terms can be rewritten as follows:

$$\underbrace{x\left(C_1+C_2+C_3\right)=xC}_{\text{"price of output"}} \neq \underbrace{xC_1+yV_1+zM_1}_{\text{"price of goods consumed during production process"}} \tag{4}$$

$$\underbrace{y\left(V_1+V_2+V_3\right)=yV}_{\text{"price of output"}} \neq \underbrace{xC_2+yV_2+zM_2}_{\text{"price of goods consumed during production process"}} \tag{5}$$

$$\underbrace{z\left(M_1+M_2+M_3\right)=zM}_{\text{"price of output"}} \neq \underbrace{xC_3+yV_3+zM_3}_{\text{"price of goods consumed during production process"}} \tag{6}$$

Adam Smith (1776) separated two senses of term "labor": "labor cost" and "labor commanded".

Passage about "labor commanded":

"Every man is rich or poor according to the degree in which he can afford to enjoy the necessaries, conveniencies, and amusements of human life. But after the division of labour has once thoroughly taken place, it is but a very small part of these with which a man's own labour can supply him. The far greater part of them he must derive from the labour of other people, and he must be rich or poor according to the quantity of that labour which he can command, or which he can afford to purchase. The value of any commodity, therefore, to the person who possesses it, and who means not to use or consume it himself, but to exchange it for other commodities, is equal to the quantity of labour which it enables him to purchase or command. Labour, therefore, is the real measure of the exchangeable value of all commodities…

Wealth, as Mr. Hobbes says, is power… The power which that possession immediately and directly conveys to him [person], is the power of purchasing; a certain **command over all the labour***, or over all the produce of labour which is then in the market. His fortune is greater or less, precisely in proportion to the extent of this power; or to the quantity either of other men's labour, or, what is the same thing, of the produce of other men's labour, which it enables him to purchase or command. The exchangeable value of every thing must always be precisely equal to the extent of this power which it conveys to its owner" (Smith (1776), v.I, ch.5).*

Passage about "labor cost":

"The real price of every thing, what every thing really **costs** *to the man who wants to acquire it, is the toil and trouble of acquiring it… What is bought with money or with goods is purchased by labour, as much as what we acquire by the toil of our own body. That money or those goods indeed save us this toil. They contain the value of a certain quantity of labour which we exchange for what is supposed at the time to contain the value of an equal quantity. Labour was the first price, the original purchase-money that was paid for all things.*
…Though labour be the real measure of the exchangeable value of all commodities, it is not that by which their value is commonly estimated. It is often difficult to ascertain the proportion between two different quantities of labour. The time spent in two different sorts of work will not always alone determine this proportion. The different degrees of hardship endured, and of ingenuity exercised, must likewise be taken into account. There may be more labour in an hour's hard work than in two hours easy business; or in an hour's application to a trade which it cost ten years labour to learn, than in a month's industry at an ordinary and obvious employment. But it is not easy to find any accurate measure either of hardship or ingenuity. In exchanging indeed the different productions of different sorts of labour for one another, some allowance is commonly made for both…" (Smith (1776), v.I, ch.5)

We will demonstrate that this distinction between "labor cost" and "labor commanded" is very important point for true understanding of transformation problem. "Labor cost" coincides with "labor commanded" only if producers are exchanging goods on the base of "values".

We will consider economy in which "non-trivial balance conditions" (NTBC) are fulfilled. Marx in chapter XX(4) of the second volume of "Capital" has introduced NTBC for the Model-2 but Model-2 can be transformed into the Model-1 by means of new definition of the second and third departments. We will discuss in details economical sense and Marx's non-trivial conditions in section VI and we will demonstrate that NTBC were carried out in early capitalist economy.

Let's formulate NTBC mathematically:

A. NON-TRIVIAL BALANCE-CONDITIONS IN THE MODEL-1:

(A1) $xC_2 = yV_1$

(A2) $xC_3 = zM_1$

(A3) $yV_3 = zM_2$

B. NON-TRIVIAL BALANCE-CONDITIONS IN THE MODEL-2:

(B1) $xC_2 = y(V_1 + M_{1V})$

(B2) $xC_3 = yM_{1m}$

(B3) $y(V_3 + M_{3V}) = zM_{2m}$

Here the following designations are used:

$V_n; C_n$ - variable and constant capital in department n;

M_{nV} - expenditures of capitalists on "consumer goods" in department n;

M_{nm} - expenditures of capitalists on "luxury goods" in department n;

M_n - the total expenditures of capitalists in department n.

Relations (A) determine equilibrium price-vector which depends on one multiplicative constant.

Non-zero solution of system (A) exists if only determinant of matrix of coefficients equals zero.

$$\begin{vmatrix} C_2 & -V_1 & 0 \\ C_3 & 0 & -M_1 \\ 0 & V_3 & -M_2 \end{vmatrix} = 0 \qquad (7)$$

It gives us the following equivalent equalities:

$$\frac{M_1}{M_2} = \frac{C_3 V_1}{C_2 V_3}; \qquad (8)$$

$$\frac{m_1}{m_2} = \frac{V_2 C_3}{V_3 C_2} = \frac{k_2}{k_3}. \qquad (9)$$

We will use the following designations:

$m = \dfrac{M}{V}$ - rate of surplus-value;

$k = \dfrac{V}{C}$ - "organic composition of the capital"[4].

Multipliers $x; y; z$ satisfy to the following relations:

$$t = \frac{x}{y} = \frac{V_1}{C_2} \qquad (10)$$

$$y = \frac{M_2 z}{V_3} \qquad (11)$$

$$x = ty = \frac{M_2 V_1 z}{V_3 C_2} \qquad (12)$$

Solution of system (A) depends on arbitrary positive value z. This is the only arbitrary variable. Other two variables y and z are defined by means of relations (11) – (12). Consequently equilibrium prices of balanced exchange follow from non-trivial balance conditions (A).

We consider two partial cases: exchange on the base of "values" and exchange on the base of "prices of production". Each case corresponds to the some "value-structure" of economy.

CASE №1. Exchange on the base of "values" – value-structure (SV);

CASE №2. Exchange on the base of "prices of production" – value-structure (SP).

[4] Although such definition of "organic composition" (V : C) differs from traditional definition (C : V) we will use the name "organic composition" for the ratio: k = V : C.

III. Value-composition of economy.

Value-structure of economy can be described by means of “labor cost” and “labor commanded” forms. “Value-composition” of economy comprises both these forms.

Table 3. Matrix III of “value-composition” for the model-1 of simple production.

	Matrix III(1). Goods consumed in each department (labor commanded aspect).				
	C (means of production) (1)	V (consumer goods) (2)	M (luxury goods) (3)	m	W (value of goods consumed in each department) (5)
I	C_1	$\beta(C-C_1)$	M_1	m_1	$C_1+\beta(C-C_1)(1+m_1)$
II	C_2	$\beta(V-C_2)$	M_2	m_2	$C_2+\beta(V-C_2)(1+m_2)$
III	C_3	$\beta(M-C_3)$	M_3	m_3	$C_3+\beta(M-C_3)(1+m_3)$
Σ	$C=C_1+C_2+C_3$	$V=\beta(V+M)$	$M=\frac{1-\beta}{\beta}V$		
	Matrix III(2). Embodied labor in product (labor costs aspect)				
	C (transferring labor)	$\widehat{V}$ (necessary labor)	$\widehat{M}$ (surplus labor)	m	$\widehat{W}$ (labor cost of product)
I	C_1	$\alpha(C-C_1)$	$(1-\alpha)(C-C_1)$	m	C
II	C_2	$\alpha(V-C_2)$	$(1-\alpha)(V-C_2)$	m	V
III	C_3	$\alpha(M-C_3)$	$(1-\alpha)(M-C_3)$	m	M
Σ	C				

“Labor values” in labor theory of value coincide with “labor costs”. “Necessary labor” is proportional to labor added during production process. Consequently the rates of surplus value in each department are equal (matrix III(2)). Employees of each industry obtain wage which is proportional to their labor. Matrix III(2) describes equilibrium state on labor-market. Labor added during production process (year) is equal $L=\widehat{V}+\widehat{M}$. Surplus-value $\widehat{M}$ in the Model-1 is equal to value of “luxury goods” consumed during year:

$$\widehat{M}=(1-\alpha)(V+M)=M \tag{13}$$

We find (from matrix III(1) and formula (13)):

$$\alpha=\beta \tag{14}$$

Matrix III describes the simple reproduction since value of product of each department (last column in matrix III(2)) is equal to value of product consumed by all departments (last raw in matrix III(1)). Rate of surplus value in matrix III(2) and parameter $0<\beta<1$ are connected by the following relations.

$$m=\frac{1-\beta}{\beta} \qquad \beta=\frac{1}{1+m} \tag{15}$$

“Rates of surplus value” in matrix III(1) can differ from the true surplus-rate (15).

$$m_n=\frac{M_n}{V_n} \tag{16}$$

These "rates" satisfy to the following formula:

$$mV = \sum_{n=1}^{3} m_n V_n \qquad (17)$$

Solution of system (A) must satisfy to relations (13)-(17).

IV. Exchange based on "values" (SV-structure).

If goods are exchanging on the base of values then we have relation:

$$x = y = z = 1 \qquad (18)$$

Balanced prices (A) coincide with prices of exchange on the base of values. "Value-structure" ("labor commanded" aspect) has the following form in this case:

Table 4. Matrix IV of value-structure for the exchange on the base of "values" (SV-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	YC	XC	$V = kC$
III.	bC	XC	ZC	M
Σ	C	$V = kC$	$M = mV$	$C+V+M$

The next formulas are fulfilled in this case:

$$\beta = \frac{V_1}{C - C_1} = \frac{(1-b)(1-a)}{1-a(1-b)} \qquad (19)$$

$$m = \frac{M}{V} = \frac{b}{(1-a)(1-b)} \qquad (20)$$

$$X = \frac{b\left[k-(1-b)(1-a)\right]}{1-a(1-b)} \qquad (21)$$

$$Y = \frac{\left[k-(1-b)(1-a)\right](1-b)(1-a)}{1-a(1-b)} \qquad (22)$$

$$Z = \frac{b^2\left[k-(1-b)(1-a)\right]}{\left[1-a(1-b)\right](1-a)(1-b)} \qquad (23)$$

$$m = m_1 = m_2 = m_3 = \frac{M}{V} = \frac{X}{Y} = \frac{Z}{X} = \frac{b}{(1-a)(1-b)} \qquad (24)$$

$$k = \frac{V}{C} \qquad (25)$$

$$k_2 = \frac{V_2}{C_2} = \frac{V_3}{C_3} = k_3 = \frac{k-(1-b)(1-a)}{1-a(1-b)} \qquad (26)$$

$$r_2 = \frac{M_2}{C_2+V_2} = \frac{M_3}{C_3+V_3} = r_3 = \frac{b\left[k-(1-a)(1-b)\right]}{(1-a)(1-b)(k+b)} \qquad (27)$$

$$r = \frac{M}{C+V} = \frac{mk}{1+k} = \frac{bk}{(1-a)(1-b)(1+k)} \tag{28}$$

Let's substitute these relations into the matrix of value-structure.

Table 5. Matrix IV(1) of value-structure ("labor commanded" aspect) for the exchange based of "values" (SV-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	$\frac{[k-(1-b)(1-a)](1-b)(1-a)}{1-a(1-b)}C$	$\frac{b[k-(1-b)(1-a)]}{1-a(1-b)}C$	$V = kC$
III.	bC	$\frac{b[k-(1-b)(1-a)]}{1-a(1-b)}C$	$\frac{b^2[k-(1-b)(1-a)]}{[1-a(1-b)](1-a)(1-b)}C$	M
Σ	C	$V = kC$	$M = mV$	

Matrix IV describes the possible value-structures which are consistent with the exchange based on "values" in the Model-1 of simple production. Any other value-structures are inconsistent with exchange based of values. Price-vector for value-structure IV is vector of values. As a rule vector of values differs from vector of prices of production.

Partial case is possible when balanced prices coincide both with values and prices of production. In this case we have the following value-structure.

Table 6. Matrix V of value-structure ("labor commanded" aspect) for the exchange in which balanced prices = values = prices of production.

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	$\frac{(1-a)^2(1-b)}{a}C$	$\frac{b(1-a)}{a}C$	$V = kC$
III.	bC	$\frac{b(1-a)}{a}C$	$\frac{b^2}{a(1-b)}C$	mV
Σ	C	$V = kC = \frac{1-a}{a}C$	$M = mV = \frac{b}{a(1-b)}C$	$C+V(1+m)$

V. Exchange based on “prices of production” (SP-structure).

Let’s find value-structures which are consistent with exchange based on the prices of production. “Non-trivial balance conditions” (A) in this case can be rewritten as follows:

(A’1) $C_2x = V_1y$

(A’2) $C_3x = r(C_1x + V_1y) = M_1z$

(A’3) $V_3y = r(C_2x + V_2y) = M_2z$

(A’4) $M_3z = r(C_3x + V_3y)$

The relation (B4) follows from (B1)-(B3) and definition of profit rate:

$$r = \frac{zM}{xC + yV} \qquad (29)$$

We have symmetric matrix of input-flows.

Table 7. Matrix VI of input-flows for the exchange based on the prices of production.

	C	V	M	Σ:
I.	C_1x	V_1y	$r(C_1x+V_1y)$	Cx
II.	C_2x	V_2y	$r(C_2x+V_2y)$	Vy
III.	C_3x	V_3y	$r(C_3x+V_3y)$	Mz
Σ:	Cx	Vy	$Mz = r(Cx+Vy)$	$C+V(1+m)$

The system (B) has nonzero solution if only the following relations are fulfilled:

$$k = \frac{C}{V} = \frac{C_1 + C_2}{V_1 + V_2} = \frac{C_3}{V_3} = k_3 \qquad (30)$$

$$r = \frac{C_3}{C_1 + C_2} = \frac{V_3}{V_1 + V_2} \qquad (31)$$

The solution of system (B) depends on one arbitrary parameter (z for example):

$$t \equiv \frac{x}{y} = \frac{V_1}{C_2} \qquad (32)$$

$$x = \frac{V_1}{C_2} y \qquad (33)$$

$$y = \frac{M_3}{r(C_3t + V_3)} z \qquad (34)$$

Value-structure (“labor commanded” aspect) has the following form in this case.

Table 8. Matrix VII of value-structure for the exchange based on prices of production.

	C	V	M	Σ
I.	$(1-b)aC$	$kd(1-b)\cdot C$	$m_1kd(1-b)C$	C
II.	$(1-b)(1-a)C$	$k(1-d)(1-b)C$	$m_2k(1-d)(1-b)C$	$V=kC$
III.	bC	kbC	m_3kbC	M
Σ	C	$V=kC$	$M=mV$	$C+V+M$

Parameters of this structure are connected by means of the following relations:

$$d=\frac{1-a(1-b)}{k+b} \tag{35}$$

$$\beta=\frac{k(1-b)}{k+b} \tag{36}$$

$$m=\frac{b(1+k)}{k(1-b)} \tag{37}$$

$$m_3=\frac{m-(1-b)\left[m_1d+m_2(1-d)\right]}{b}=m \tag{38}$$

$$\frac{m_2}{m_1}=\frac{(k+b)(1-a)}{k-(1-a)(1-b)} \tag{39}$$

$$t=\frac{kd}{1-a} \tag{40}$$

$$m_1=\left(\frac{b}{1-b}\right)\cdot\frac{(k+b)(1+k)}{k\left[b+(1-a)(1+k)\right]} \tag{41}$$

$$r=\frac{b}{1-b} \tag{42}$$

Let's substitute these formulas into matrix VII.

Table 9. Matrix VII(1) of value-structure for the exchange based on prices of production (SP-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$\frac{k(1-b)\left[1-a(1-b)\right]}{k+b}\cdot C$	$\frac{b(1+k)\left[1-a(1-b)\right]}{\left[b+(1-a)(1+k)\right]}C$	C
II.	$(1-b)(1-a)C$	$\frac{k(1-b)\left[k-(1-a)(1-b)\right]}{k+b}C$	$\frac{b(1-a)(1+k)(k+b)}{b+(1-a)(1+k)}C$	$V=kC$
III.	bC	kbC	$\frac{b^2(1+k)}{1-b}C$	M
Σ	C	$V=kC$	$M=mV$	

We see that SP-structure differs from SV-structure. These structures coincide if only value-structure has the form of matrix V.

Let's list Marx's Rules of Transformation:

Rule I. Aggregate surplus-value is equal to aggregate profit:

(RI) $M = r\left(Cx + Vy\right)$

Rule 2. Value of gross output is equal to price of production of gross output.

(RII) $C + V + M = \left(1+r\right)\left(Cx + Vy\right)$

Rule III. Rates of profit calculated on the base of values and prices of production are equal.

(RIII) $\dfrac{M}{C+V} = \dfrac{Mz}{Cx+Vy}$

Let $z = 1$. The rule (RIII) follows from (RI) and (RII) in this case.

Let's prove that both first and second rules of transformation are fulfilled simultaneously for SP-structure. Substitute relations (35) – (41) into formulas (32) – (34).

$$y = \frac{\left(1+k\right)\left(1-a\right)\left(k+b\right)}{k\left[1-a+b+k\left(1-a\right)\right]} \tag{43}$$

$$t = \frac{k\left[1-a\left(1-b\right)\right]}{\left(k+b\right)\left(1-a\right)} \tag{44}$$

$$x = ty = \frac{\left(1+k\right)\left[1-a\left(1-b\right)\right]}{1-a+b+k\left(1-a\right)} \tag{45}$$

Rules (RI)-(RIII) follow from (43)-(45), (37) and (42):

$$Cx + Vy = C\left(x + ky\right) = C\left(\frac{\left(1+k\right)\left[1-a\left(1-b\right)+\left(1-a\right)\left(k+b\right)\right]}{1-a+b+k\left(1-a\right)}\right) = C\left(1+k\right) = C+V \tag{46}$$

$$r\left(Cx+Vy\right) = r\left(C+V\right) = \frac{b\left(1+k\right)C}{1-b} \tag{47}$$

$$M = mV = mkC = \frac{b\left(1+k\right)kC}{k\left(1-b\right)} = \frac{b\left(1+k\right)C}{1-b} = r\left(Cx+Vy\right) \tag{48}$$

We obtain the following structure for the economy of simple production in prices of production after substitution of formulas (43) and (45) into Table 7.

Table 9.1. Structure of economy with simple production and the exchange on the base of "production prices". Model-1.

	C (in "prices of production")	V (in "prices of production")	M (in "prices of production")
I.	$\frac{\left(1-b\right)a\left(1+k\right)\left[1-a\left(1-b\right)\right]}{1-a+b+k\left(1-a\right)}C$	$\frac{\left(1-b\right)\left[1-a\left(1-b\right)\right]\left(1+k\right)\left(1-a\right)}{\left[1-a+b+k\left(1-a\right)\right]}C$	$\frac{b\left(1+k\right)\left[1-a\left(1-b\right)\right]}{\left[b+\left(1-a\right)\left(1+k\right)\right]}C$
II.	$\frac{\left(1-b\right)\left(1-a\right)\left(1+k\right)\left[1-a\left(1-b\right)\right]}{1-a+b+k\left(1-a\right)}C$	$\frac{\left(1-b\right)\left[k-\left(1-a\right)\left(1-b\right)\right]\left(1+k\right)\left(1-a\right)}{\left[1-a+b+k\left(1-a\right)\right]}C$	$\frac{b\left(1-a\right)\left(1+k\right)\left(k+b\right)}{b+\left(1-a\right)\left(1+k\right)}C$
III.	$\frac{b\left(1+k\right)\left[1-a\left(1-b\right)\right]}{1-a+b+k\left(1-a\right)}C$	$\frac{b\left(1+k\right)\left(1-a\right)\left(k+b\right)}{\left[1-a+b+k\left(1-a\right)\right]}C$	$\frac{b^2\left(1+k\right)}{1-b}C$

VI. Non-trivial conditions of balanced exchange. Discussion.

So, Marx's transformation rules (RI)-(RIII) are fulfilled in the Model-1 if we take into account non-trivial balance conditions (B). Matrix of input-flows has symmetric form in this case. Let's demonstrate that balance conditions (A) (or (B) if exchange of goods is based on prices of production) follow from Marx's analysis of simple reproduction in the second volume of "Capital", section XX(IV).

Let's consider this point in details. Marx considers model-2: I – "means of production", IIa – "consumer necessities" ("necessities of life"), and IIb – "luxury goods". Working-class bye only consumer necessities whereas capitalists buy both consumer necessities and luxury goods. Income of working-class is a wage. Income of capitalists is profit. So, Marx's model is our Model-2. Marx considers the following numerical example.

$$\begin{aligned} &\text{II}\,a:\ 400_V + 400_m \\ &\text{II}\,b:\ 100_V + 100_m \end{aligned} \qquad (49)$$

He writes:

"The labourers of IIb have received 100 in money as payment for their labour-power, or say £100. With this money they buy articles of consumption from capitalists IIa to the same amount. This class of capitalists buys with the same money £100 worth of the IIb commodities, and in this way the variable capital of capitalists IIb flows back to them in the form of money.

In IIa there are available once more 400_v in money, in the hands of the capitalists, obtained by exchange with their own labourers. Besides, a fourth of the part of the product representing surplus-value has been transferred to the labourers of IIb, and in exchange IIb (100_v) have been received in the form of articles of luxury".

Consequently capitalists IIa bought luxury goods in sum £100. They obtained these money owing to sale in sum £100 of own product (consumer necessities) to "labourers" IIb. We see that value of consumer necessities bought by IIb is equal to value of luxury goods bought by IIa.

"Now, assuming that the capitalists of IIa and IIb divide the expenditure of their revenue in the same proportion between necessities of life and luxuries — three-fifths for necessities for instance and two-fifths for luxuries — the capitalists of sub-class IIa will spend three-fifths of their revenue from surplus-value, amounting to 400_s, or 240, for their own products, necessities of life, and two-fifths, or 160, for articles of luxury. The capitalists of sub-class IIb will divide their surplus-value of 100_s in the same way: three-fifths, or 60, for necessities, and two-fifths, or 40, for articles of luxury, the latter being produced and exchanged in their own sub-class".

So, the following Schema I illustrates results of exchange between IIa and IIb.

Marx writes:

"That in the case of annual product which is reproduced in the form of articles of consumption, the variable capital v advanced in the form of money can be realised by its recipients, inasmuch as they are labourers producing luxuries, only in that portion of the necessities of life which embodies for their capitalist producers prima facie their surplus-value; hence that ***v, laid out in the production of luxuries, is equal in value to a corresponding portion of s produced in the form of necessities of life****, and hence must be smaller than the whole of this s, namely $(IIa)_s$, and that the variable capital advanced by the capitalist producers of luxuries returns to them in the form of money only by means of the realisation of that v in this portion of s. This phenomenon is quite analogous to the realisation of $I_{(v+s)}$ in II_c, except that in the second case $(IIb)_v$ realizes*

itself in a part *of (IIa)$_s$ of the same value. These proportions remain qualitatively determinant in every distribution of the total annual product, since it actually enters into the process of the annual reproduction brought about by circulation".*

Schema I. Input-flows of goods after circulation (exchange) between sub-divisions IIa and IIb.

$$\text{II}\,a:\ \underbrace{\underbrace{400_V}_{\text{demand for "necessities of life" of labourers IIa}}}_{\text{necessities of life}} + \left(\underbrace{\underbrace{100_m}_{\text{demand for "necessities of life" of labourers IIb}} + \underbrace{60_m}_{\text{demand for "necessities of life" of capitalists IIb}}}_{\text{LUXURY GOODS (after circulation)}} + \underbrace{\underbrace{240_m}_{\text{demand for "necessities of life" of capitalists IIa}}}_{\text{necessities of life}}\right)$$

$$\text{II}\,b:\ \underbrace{\underbrace{100_V}_{\text{demand for "luxury goods" of capitalists IIa}}}_{\text{necessities of life for laboriers IIb}} + \left(\underbrace{\underbrace{60_m}_{\text{demand for "luxury" of capitalists IIb}}}_{\text{necessities of life for capitalists IIb}} + \underbrace{\underbrace{40_m}_{\text{demand for "luxury" of capitalists IIb}}}_{\text{LUXURY GOODS}}\right)$$

Marx indicates that value of input-flow of "luxury" in sub-division IIa is equal to value of input-flow of "necessities of life" in sub-division IIb (160 = 160). Goods of sub-division IIa ("necessities of life") are exchanged onto the goods of sub-division IIb ("luxury") on the base of equilibrium price of balanced exchange (value = price of production in Marx's example). Marx's rule of balanced exchange between sub-divisions IIa and IIb means that price of goods exchanged one another ("luxury" for "necessities of life") is the same. This rule in the Model-1 (where capitalists buy only "luxury") can be formulated as balance-condition (A3)[5]:

(A3) $zM_2 = yV_3$

Marx's numerical example including department I has the following Schema II.

Schema II. Marx's numerical example.

I. $4000_C + 1000_V + 1000_m$ - means of production

IIa. $1600_C + 400_V + 400_m$ - necessities of life

IIb. $400_C + 100_V + 100_m$ - luxury

Capitalists of sub-division IIa buy "luxury" in sum 160. Capitalists and laborers of subdivision IIb buy "necessities of life" in sum 160 also. Capitalists IIa consume own product ("necessities of life") in sum 240. Capitalists IIb consume own product ("luxury") in sum 40. We have the following Schema.

Schema III. Realization of profit in sub-divisions IIa and IIb.

$$400_m\left(\text{IIa}\right) = \underbrace{160_m}_{\text{LUXURY}} + \underbrace{240_m}_{\text{NECESSITIES OF LIFE}}$$

$$100_m\left(\text{IIb}\right) = \underbrace{40_m}_{\text{LUXURY}} + \underbrace{60_m}_{\text{NECESSITIES OF LIFE}}$$

Let's substitute this composition into Schema II.

[5] Balance-conditions (A1) and (A2) follow from (A1) in the economy with simple reproduction.

Schema IV. Matrix of input-flows.

$$\text{I.}\quad 4000_C + 1000_V + 1000_m \text{ - means of production}$$

$$\text{IIa.}\ 1600_C + \underbrace{400_V}_{\text{NECESSITIES OF LIFE}} + \left(\underbrace{240_m}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \right) \text{ - necessities of life}$$

$$\text{IIb.}\ 400_C + \underbrace{100_V}_{\text{NECESSITIES OF LIFE}} + \left(\underbrace{60_m}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \right) \text{ - luxury}$$

Value of "luxury" consumed in sub-division IIa is equal to value of "necessities of life" consumed in sub-division IIb. Rearrange Schema IV as follows:

Schema V. Matrix of input-flows.

$$\text{I.}\quad 4000_C + 1000_V + 1000_m \text{ - means of production}$$

$$\text{IIa.}\ 1600_C + \underbrace{640_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \text{ - necessities of life}$$

$$\text{IIb.}\ 400_C + \underbrace{160_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \text{ - luxury}$$

Exchange between sub-division IIa and department I is analogous. Capitalists IIa buy "means of production" in sum 1600. Capitalists and laborers I buy "necessities of life" in sum 1600 (1000 – laborers I and 600 – capitalists I). Capitalists IIb buy "means of production" in sum 400. Capitalists I buy "luxury" in the same sum.
Finally, we have the following complete Schema of input-flows for Marx's example.

Schema VI. Matrix of input-flows in Marx's numerical example.

$$\text{I.}\quad \underbrace{4000_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{1600_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{400_m}_{\text{LUXURY}} \text{ - means of production}$$

$$\text{IIa.}\quad \underbrace{1600_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{640_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \text{ - necessities of life}$$

$$\text{IIb.}\quad \underbrace{400_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{160_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \text{ - luxury}$$

We see that all balance-conditions (A) are fulfilled. Matrix of input-flows in prices of balanced exchange is symmetric.

Revenue of capitalists in general case consists of manager's wage and net profit as proprietor of capital. We assumed that capitalists are spent wage on "necessities of life" and they are spent net profit on "luxury goods" only. Marx's Model-2 can be transformed into Model-1 if we assume that the second column in Schema V consists of goods which were bought on wage of workers and wage of capitalists.

Marx emphasized that concrete figures and proportions in his numerical example can be chosen arbitrary. The general logical conclusions don't depend on the concrete figures in this numerical example:

What is arbitrary here is the ratio of the variable to the constant capital of both I and II and so is the identity of this ratio for I and II and their sub-divisions. As for this identity, it has been assumed here merely for the sake of simplification, and it would not alter in any way the conditions of the problem and its solution if *we were to assume different proportions. However, the necessary result of all this, on the assumption of simple reproduction, is the following…*

In the exchange established above of $(IIb)_v$ for a portion of $(IIa)_s$ of the same value, and in the further exchanges between (IIa), and (IIb), it is by no means assumed that either the individual capitalists of IIa and IIb or their respective totalities divide their surplus-value in the same proportion between necessary articles of consumption and articles of luxury. The one may spend more on this consumption, the other more on that. On the basis of simple reproduction it is merely assumed that a sum of values equal to the entire surplus-value is realised in the consumption-fund. The limits are thus given. Within each department the one may spend more in a, the other in b. (vol. II; chapter XX(4))

Marx's numerical example corresponds to early capitalist economy in which credit plays very minor role. Marx writes:

"It goes without saying that this applies only to the extent that it all is really a result of the process of reproduction itself, i.e., to the extent that the capitalists of IIb, for instance, do not obtain money-capital for v on credit from others" (vol. II; chapter XX(4))

"…credit-production plays only a very minor role, or none at all, during the first epoch of capitalist production" (vol. II; chapter IV)

Why Marx postulated non-trivial conditions of exchange in the early capitalist economy with simple production? The answer in details is given in paper Pushnoi (2011) where the Model-4 is considered. We may list here (in this paper, where only model-1 and model-2 are considered) only some historical and logical arguments.

Argument I. The models -1 and 2 don't take into account "labor of capitalists" as entrepreneurs although this "labor" enters into "values" of goods like to labor of workers. Model-4 is more realistic model of capitalist simple production since this model takes into account "labor of capitalists". Solution of transformation problem in this model exists under any value-structure of economy. Marx's non-trivial balance-conditions are fulfilled automatically if (1) "luxury per subsistence goods" proportion in surplus-value α is same in each department ($\alpha_1 = \alpha_2 = \alpha_3$) and if (2) ratio of "capitalist salary" (the payment for the labor of "entrepreneur") to value of variable capital γ is the same in all departments ($\gamma_1 = \gamma_2 = \gamma_3$).

Argument II. It is very probable that conditions $\alpha_1 = \alpha_2 = \alpha_3$ and $\gamma_1 = \gamma_2 = \gamma_3$ were carried out in the early capitalist economy of merchant capitalism.

"In the precapitalist stages of society, commerce rules industry. The reverse is true of modern society. Of course, commerce will have more or less of a reaction on the societies, between which it is carried on. It will subject production more and more to exchange value, by making enjoyments and subsistence more dependent on the sale than on the immediate use of the products…

First, the merchant becomes directly an industrial capitalist. This is the case in crafts conditioned on commerce, especially industries producing luxuries, which are imported by the merchants together with the raw materials and laborers from foreign countries, as they were in Italy from Constantinople in the 15th century. In the second place, the merchant converts the small masters into his middlemen or, perhaps, buys direct from the self-producer, leaving him nominally independent and his mode of production unchanged. In the third place, the industrial becomes a merchant and produces immediately on a large scale for commerce…

The merchant becomes an industrial capitalist, or rather, he lets the craftsmen, particularly the small rural producers, work for him." (Marx; "Capital"; v. III, chap.XX).

Merchants invested money-capital in different projects: both into production of means of production and into production of subsistence goods and luxury. Therefore the capital of each department consisted of separate capitals of many merchants. Although parameters α and γ for each merchant-capitalist differed one another the average values of parameters in the departments as a whole were almost equal ("the law of big numbers"). Often the capital of separate merchant was dispersed between three departments of economy. Each merchant union (or separate merchant) participated often in production of different products: "means of production"; "subsistence goods" and "luxury goods". "The law of big numbers" was equalizing parameters α and γ of different departments. Conditions $\alpha_1 = \alpha_2 = \alpha_3$ and $\gamma_1 = \gamma_2 = \gamma_3$ guarantee the execution of non-trivial balance-conditions in the model-4. Consequently we can conclude that non-trivial balance-conditions were executed in epoch of early capitalism.

VII. Bortkiewicz' solution.

Bortkiewicz (1907a) considered Model-1. Let's remember his problem statement.
He postulated the following relations[6]:

1) Trivial conditions of simple production:

$$\begin{aligned} C_1 + C_2 + C_3 &= C_1 + V_1 + M_1 \\ V_1 + V_2 + V_3 &= C_2 + V_2 + M_2 \\ M_1 + M_2 + M_3 &= C_3 + V_3 + M_3 \end{aligned} \tag{50}$$

2) Conditions for value-structure of economy:

$$\begin{aligned} C_1 + V_1 + M_1 &= C \\ C_2 + V_2 + M_2 &= V \\ C_3 + V_3 + M_3 &= M \end{aligned} \tag{51}$$

3) Conditions for surplus-value:

$$\begin{aligned} M_1 &= mV_1 \\ M_2 &= mV_2 \\ M_3 &= mV_3 \end{aligned} \tag{52}$$

Relations (52) are fulfilled in "labor-costs"-structure only. Consequently M_n is surplus-value. Relations (50) – (52) describes "labor-costs"-structure in the economy with simple production. Symbols $M_1;M_2;M_3$ does not designate value of something goods. These are symbols which fix surplus-value in each department. We demonstrated earlier that value of goods consumed by capitalists of each department can differ from surplus-value of each department. These quantities coincide if only exchange of goods based on values. Consequently symbols $M_1;M_2;M_3$ don't mean any goods – "luxury" for example. These symbols designate surplus-value only. Bortkiewicz does not distinct "labor cost" and "labor commanded" aspects. As consequence he identified surplus-value of each department and value of goods consumed by capitalists of this department. Moreover non-trivial balance-conditions were executed in early capitalist economy. Let's correct Bortkiewicz' relations (50) – (52) as follows:

1) Conditions of simple production:

$$\begin{aligned} C_1 + C_2 + C_3 &= C \\ V_1 + V_2 + V_3 &= V \\ \widehat{M}_1 + \widehat{M}_2 + \widehat{M}_3 &= \widehat{M} \end{aligned} \tag{50c}$$

[6] Bortkiewicz used designation S for surplus-value.

2) Conditions for value-structure of economy:

$$C_1 + V_1 + \widehat{M}_1 = C$$
$$C_2 + V_2 + \widehat{M}_2 = V \qquad (51c)$$
$$C_3 + V_3 + \widehat{M}_3 = \widehat{M}$$

3) Conditions for surplus-values:

$$\widehat{M}_1 = mV_1$$
$$\widehat{M}_2 = mV_2 \qquad (52c)$$
$$\widehat{M}_3 = mV_3$$

Symbols $\widehat{M}_n$ designate surplus-value. It is necessary to take into account relations for "labor commanded" - structure in order to obtain complete problem statement. These are the following relations:

$$\underbrace{\left(\underbrace{M}_{\text{value of "luxury"}} = \underbrace{M_1}_{\text{"luxury" consumed in department I}} + \underbrace{M_2}_{\text{"luxury" consumed in department II}} + \underbrace{M_3}_{\text{"luxury" consumed in department III}} \right)}_{\text{"LABOR COMMANDED"}} = \underbrace{\left(\underbrace{\widehat{M}}_{\text{surplus-value}} = \underbrace{\widehat{M}_1}_{\text{surplus-value produced in department I}} + \underbrace{\widehat{M}_1}_{\text{surplus-value produced in department II}} + \underbrace{\widehat{M}_1}_{\text{surplus-value produced in department III}} \right)}_{\text{"LABOR COSTS"}} \qquad (53)$$

$$\widehat{M} = mV = M = \sum_{n=1}^{3} M_n = \sum_{n=1}^{3} m_n V_n \qquad (54)$$

$$\frac{m_1}{m_2} = \frac{V_2 C_3}{V_3 C_2} = \frac{k_2}{k_3} \qquad (55)$$

Bortkiewicz has not taken into account the "labor-commanded-aspect" of value-composition and non-trivial balance-conditions in early capitalist economy. Consequently Bortkiewicz' problem statement was not complete.

Let's consider Bortkiewicz' equations for "production prices":

$$(1+r)(C_1 x + V_1 y) = xC$$
$$(1+r)(C_2 x + V_2 y) = yV \qquad (56)$$
$$(1+r)(C_3 x + V_3 y) = zM = zM_1 + zM_2 + zM_3$$

Let's impose yet non-trivial balance-conditions in the form (B) (for the exchange based on "production prices"):

(B1) $C_2 x = V_1 y$

(B2) $C_3 x = r(C_1 x + V_1 y) = M_1 z$

(B3) $V_3 y = r(C_2 x + V_2 y) = M_2 z$

(B4) $M_3 z = r(C_3 x + V_3 y)$

These are Marx's non-trivial balance-conditions for the model-1 with the exchange based on "prices of production". Relations (B) should be carried out in early capitalist economy with exchange based on "production prices". We illustrated in previous Section that relations (B) follow from Marx's schemas of simple reproduction. These relations impose condition of symmetry onto "input-flows matrix" of Model-1.

Bortkiewicz didn't take into account these non-trivial balance-conditions. Mathematically it means that he spread the set of possible solutions including even such solutions which don't satisfy to conditions (B). Consequently Bortkiewicz considered more wide set of "solutions". We proved (formulas (46)-(48)) that solution of transformation problem (for the economy of simple production with exchange based on "production prices" and non-trivial balance-conditions) exists always in the Model-1 (and in the Model-2 also – see below).

So, Bortkiewicz' solution doesn't take into account two aspects of task. First, Bortkiewicz omitted analysis of "labor commanded" aspect i.e. "labor-structure" which describes the proportions by which each department consumes the different articles of product produced by all departments. Second, Bortkiewicz omitted non-trivial balance-conditions (B) which follow from Marx's analysis of schemas of simple reproduction in the second volume of "Capital".

VIII. The problem of historical transformation.

We proved that Marx's rules of transformation are fulfilled in the Model-1 if exchange of goods based on "prices of production". Marx assumed that goods exchanged as "values" in the past (in pre-capitalist era). Each "value-structure" is connected with the definite equilibrium price-vector of balanced exchange. This price-vector coincides with values if "value-structure" of economy has the form of matrix IV (SV-structure) and it coincides with "production prices" if "value-structure" has the form of matrix VII (SP-structure). Consequently the process of **historical** transformation could take place if only value-structure (SV) transformed into value-structure (SP) during some "transition period". How such process could proceed?

Let's introduce "technological coefficients" γ_n as follows:

$$\gamma_1 = \frac{C_1}{C}; \qquad C_1 = C\gamma_1 \tag{57}$$

$$\gamma_2 = \frac{C_2}{V} \qquad C_2 = V\gamma_2 \tag{58}$$

$$\gamma_3 = \frac{C_3}{M} \qquad C_3 = M\gamma_3 \tag{59}$$

These quantities depend only on technologies of production in each department. The change of these coefficients indicates on change in technique and organization of labor. Value-composition (Table III) and some calculations give us the following relations:

$$\beta = \frac{1}{1+m} \tag{60}$$

$$V_1 = \frac{1-\gamma_1}{1+m}C \tag{61}$$

$$V_2 = \frac{1-\gamma_2}{1+m}V \tag{62}$$

$$V_3 = \frac{1-\gamma_3}{1+m}M \tag{63}$$

$$k_n = \frac{1-\gamma_n}{\gamma_n(1+m)} \qquad n = 1;2;3 \tag{64}$$

$$k = \frac{1-\gamma_1}{\gamma_2 + m\gamma_3} \tag{65}$$

$$r_n = \frac{m_n V_n}{C_n + V_n} = \frac{m_n(1-\gamma_n)}{1+\gamma_n m} \quad (66)$$

$$q_1 \equiv \frac{V_1}{V} = \frac{1-\gamma_1}{k(1+m)} \quad (67)$$

$$q_2 \equiv \frac{V_2}{V} = \frac{1-\gamma_2}{1+m} \quad (68)$$

$$q_3 \equiv \frac{V_3}{V} = \frac{m(1-\gamma_3)}{1+m} \quad (69)$$

Matrix of value-structure can be rewritten in new variables as follows:

Table 10. Matrix VIII of value-structure via technological coefficients.

	C	V	M	Σ
I.	$\gamma_1 C$	$\beta(1-\gamma_1)\cdot C$	$m_1\beta(1-\gamma_1)C$	C
II.	$\gamma_2 kC$	$\beta(1-\gamma_2)kC$	$m_2\beta(1-\gamma_2)kC$	$V = kC$
III.	$\gamma_3 mkC$	$\beta(1-\gamma_3)mkC$	$m_3\beta(1-\gamma_3)mkC$	$M = mkC$
Σ	C	$V = kC$	$M = mV = mkC$	

Let's consider three cases.

CASE №1. Exchange based on values ($x = y = z$) gives the following equalities:

$$\gamma_2 = \gamma_3 \quad (70)$$

$$m = m_1 = m_2 = m_3 \quad (71)$$

$$k_2 = k_3 \quad (72)$$

CASE №2. "Transition period". The next relations follow from conditions (A):

$$\frac{m_1}{m_2} = \frac{k_2}{k_3} = \frac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)} \quad (73)$$

$$m = \frac{1-\beta}{\beta} \quad (74)$$

$$m = q_1 m_1 + q_2 m_2 + q_3 m_3 \quad (75)$$

Conditions (73) – (75) determine the set of possible value-structures during "transition period". Other quantities are equal:

$$k = \frac{1-\gamma_1}{\gamma_2 + m\gamma_3} \quad (76)$$

$$m_2 = \frac{m\left[1+m-m_3(1-\gamma_3)\right]k\gamma_2(1-\gamma_3)}{(1-\gamma_2)\left[\gamma_3(1-\gamma_1)+k\gamma_2(1-\gamma_3)\right]} = \frac{m\left[1+m-m_3(1-\gamma_3)\right]\gamma_2(1-\gamma_3)}{(1-\gamma_2)\left[\gamma_2+m\gamma_3^2\right]} \quad (77)$$

$$m_1 = \frac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)} m_2 \quad (78)$$

Formulas (70) – (72) is partial case of general relations (73) – (78).

CASE №3. Exchange based on "production prices".
We have in this case the following relations:

$$\frac{C}{C_1+C_2}=\frac{V}{V_1+V_2}=\frac{M}{M_1+M_2}=1+r \quad (79)$$

These relations give the following equalities:

$$k_3=k \quad (80)$$

$$m_3=m \quad (81)$$

Relations (80) – (81) are fulfilled if rate of surplus-value is equal:

$$m=\frac{\gamma_2(1-\gamma_3)-\gamma_3(1-\gamma_1)}{\gamma_3(\gamma_3-\gamma_1)}=\frac{\frac{\gamma_2}{\gamma_3}-\chi}{\chi-1} \quad (82)$$

$$\chi=\frac{1-\gamma_1}{1-\gamma_3} \quad (83)$$

Consequently all parameters in matrix VIII are functions of three technological coefficients $\gamma_1;\gamma_2;\gamma_3$. Parameters of matrices VII and VIII are interconnected as follows:

$$\gamma_1=a(1-b) \quad (84)$$

$$\gamma_2=\frac{(1-a)(1-b)}{k} \quad (85)$$

$$\gamma_3=\frac{1-b}{1+k} \quad (86)$$

$$a=\frac{\gamma_1(\gamma_2-\gamma_3)}{\gamma_1(\gamma_2-\gamma_3)+\gamma_2(\gamma_3-\gamma_1)} \quad (87)$$

$$b=\frac{\gamma_2(1-\gamma_3)-\gamma_3(1-\gamma_1)}{\gamma_2-\gamma_3} \quad (88)$$

$$k=\frac{\gamma_3-\gamma_1}{\gamma_2-\gamma_3} \quad (89)$$

We see that initial exchange based on values supposes other value-structure than exchange based on production prices. (SV) and (SP) structures coincide when these value-structures has the same form of matrix V (Table V).

Process of historical transformation can be described theoretically as some "trajectory" of the economic system in 4-dimensional space $(m;\gamma_1;\gamma_2;\gamma_3)$. Two variants of historical transformation are possible theoretically in general case as it follows from formulas (82) – (83). These variants describe economy with positive rate of surplus-value. Technological coefficients after transformation must satisfy to one of two systems of inequalities:

VARIANT №1.

$$\gamma_1<\gamma_2 \quad (90)$$

$$\gamma_1<\gamma_3<\frac{\gamma_2}{1+(\gamma_2-\gamma_1)} \quad (91)$$

VARIANT №2.

$$\gamma_1>\gamma_2 \quad (92)$$

$$\gamma_1 > \gamma_3 > \frac{\gamma_2}{1+(\gamma_2-\gamma_1)} \tag{93}$$

We have conditions $\gamma_2 = \gamma_3$ and $m_3 = m$ for initial state of economy (Case №1). Transformation is possible if coefficients γ_n after transformation satisfy to inequalities (90) – (91) or to inequalities (92) – (93). Condition $m_3 = m$ is fulfilled both before transformation and after transformation. Let's assume that this condition is fulfilled also during transition period: $m_3 = m$

There is some kind of a "barrier" which separates region in the space $(m;\gamma_1;\gamma_2;\gamma_3)$ where exchange based on production prices is possible. Relative change of parameter γ_3 must be larger than this "barrier":

$$\left|\frac{\Delta\gamma_3}{\gamma_3}\right| \geq \left|\frac{\gamma_2-\gamma_1}{1+(\gamma_2-\gamma_1)}\right| \tag{94}$$

Figure 1 illustrates position of this barrier in the plane $(\gamma_2;\gamma_3)$.

Balanced exchange in transition period differs both from exchange based on values and exchange based on prices of production. The following formulas for profit rates can be deduced from conditions (A):

$$1+r_1 = \frac{Cx}{C_1x+V_1y} = \frac{Cx}{C_1x+C_2x} = \frac{C}{C_1+C_2} \tag{95}$$

$$1+r_2 = \frac{Vy}{C_2x+V_2y} = \frac{Vy}{V_1y+V_2y} = \frac{V}{V_1+V_2} \tag{96}$$

$$1+r_3 = \frac{Mz}{C_3x+V_3y} = \frac{Mz}{M_1z+M_2z} = \frac{M}{M_1+M_2} \tag{97}$$

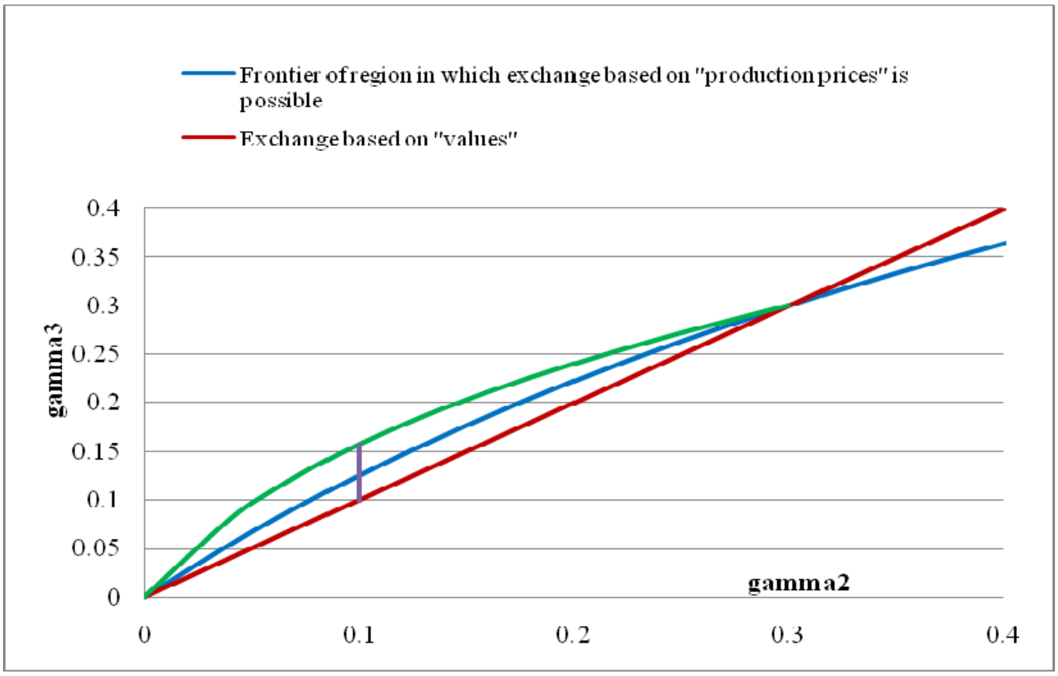
It gives us (accounting formulas (76) – (78) and (93)) the following relations:

$$r_1 = \frac{m\gamma_3(1-\gamma_1)}{\gamma_2+m\gamma_1\gamma_3} \tag{98}$$

$$r_2 = \frac{m(1-\gamma_3)}{1+m\gamma_3} \tag{99}$$

$$r_3 = \frac{m(1-\gamma_3)}{1+m\gamma_3} \tag{100}$$

Figure 1. Region of exchange based on production prices in the plane of technological coefficients $(\gamma_2;\gamma_3)$.



Historical transformation couldn't contradict to the aspiration of capitalists to increase the profit rate. We see from formulas (98) – (100) that rates $r_3;r_2$ decreases when technological coefficient γ_3 grows. Coefficient γ_3 is the share of means of production in value of product. Technological progress leads to the increase of this share (= economy of living labor). Therefore coefficients γ_n (and consequently rate of surplus-value as it follows from formulas (99) – (100)) should increase in the process of historical transformation.

We modeled this process for two variants of transformation (Variant №1 and Variant №2). Our model of historical transformation assumes that coefficients $\gamma_1;\gamma_2$ don't change whereas coefficients $\gamma_1;m$ can change. This is the model of historical transformation on account of technical progress in third department ("luxury goods"). Let γ_{3P} be new parameter γ_3 and m_P be new rate of surplus-value after transformation. We imposed condition $r_3 = Const$ during transition period (technical progress shouldn't decrease the rate of profit in modernized department). Formulas (98) – (100) and condition $r = r_1 = r_2 = r_3$ after transformation lead to the following relations:

$$\gamma_{3P} = \frac{-B+\sqrt{B^2-4AC}}{2A} \tag{101}$$

$$A = m(1-\gamma_3)-(1+m)(1+\gamma_2-\gamma_1) \tag{102}$$

$$B = \gamma_2(1+m)+(1+m\gamma_3)(1+\gamma_2-\gamma_1)-\gamma_1 m(1-\gamma_3) \tag{103}$$

$$C = -\gamma_2(1+m\gamma_3) \tag{104}$$

$$m_P = \frac{m(1-\gamma_3)}{(1-\gamma_{3P})+m(\gamma_3-\gamma_{3P})} \tag{105}$$

$$r_{1P} = \frac{m_P\gamma_{3P}(1-\gamma_1)}{\gamma_2 + m_P\gamma_1\gamma_{3P}} = r_{2P} = r_{3P} = \frac{m_P(1-\gamma_{3P})}{1+m_P\gamma_{3P}} = r_3 = \frac{m(1-\gamma_3)}{1+m\gamma_3} \quad (106)$$

Figures 2 - 4 illustrate the growth of rate of surplus-value after transformation. We see that the increase m_P is moderate if $\gamma_2 = \gamma_3 < 0.4$ before transformation. We suppose that such figures are in the consistence with our knowledge about economy of pre-capitalist era in which level of technical development was very low. Significant changes in rate of surplus-value during transition period were practically impossible because both capitalists and workers struggle against any change of this rate. The Supplement 2 contains description of two models (VARIANT №1 and VARIANT №2).

Figure 2. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).

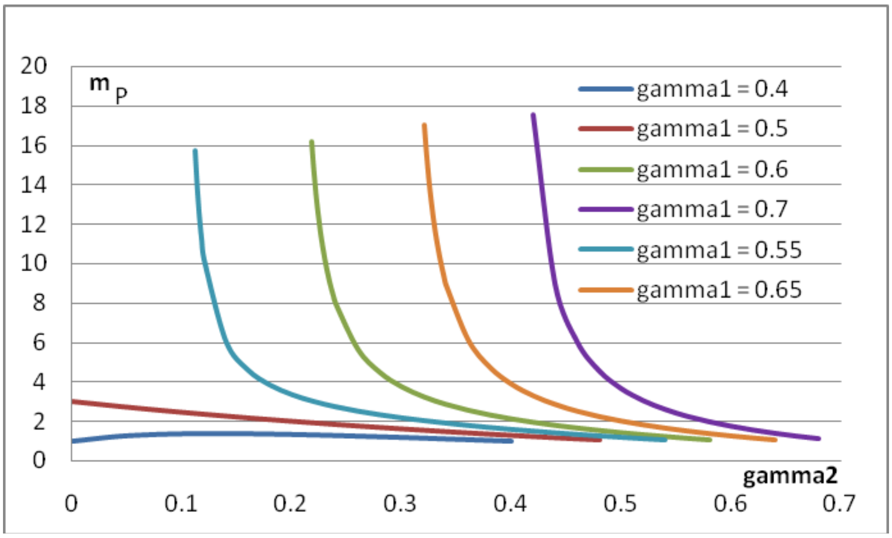


Figure 3. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).

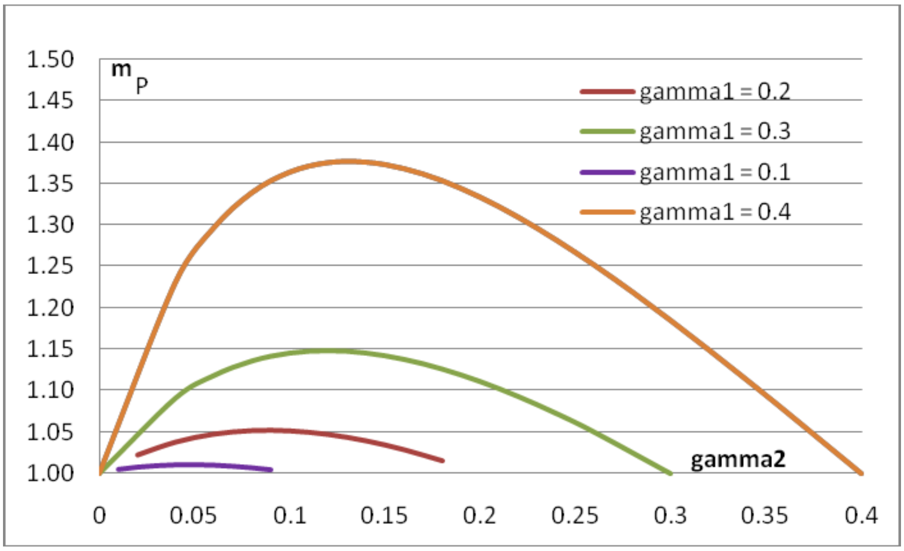
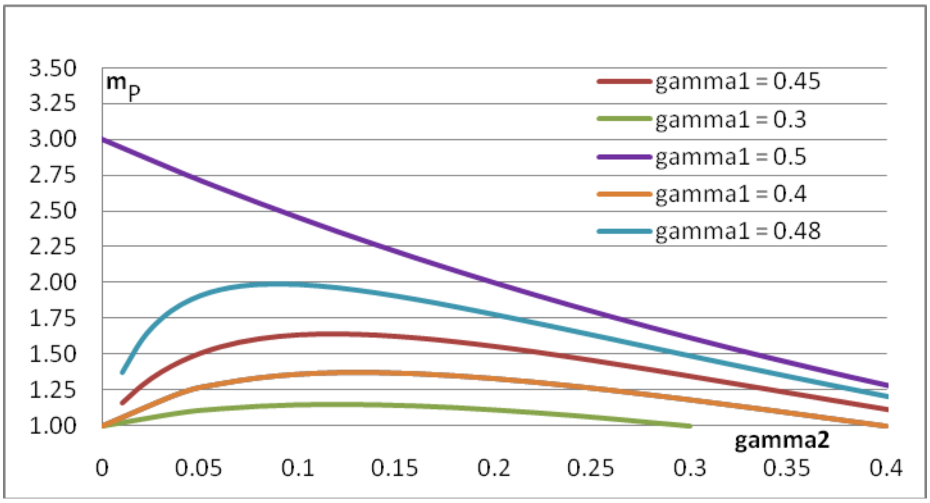


Figure 4. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).



IX. Problem of current transformation of "values" into "prices" in the model-2.

Consider the general case when matrix of input flows is asymmetric. This is case when profit of capitalists is spent on both "luxury" and "consumer goods". Structure of economy in this case has the following form:

Table XI. Matrix of input-flows.

	C	V	M	W
I	xC_1	yV_1	$yM_{1V}+zM_{1m}$	xC
II	xC_2	yV_2	$yM_{2V}+zM_{2m}$	$y\left(V+M_V\right)$
III	xC_3	yV_3	$yM_{3V}+zM_{3m}$	zM_m
Σ	xC	yV	yM_V+zM_m	

Symbols M_V (M_m) designate value of "consumer goods" ("luxury") consumed by capitalists. Multipliers $(x;y;z)$ transfer "values" into "prices of production". The aggregates ($yM_{1V}+zM_{1m}$ and xC_3); ($yM_{2V}+zM_{2m}$ and yV_3) are not equal in this case (asymmetric matrix of input-flows).

Non-trivial conditions of balanced exchange for the simple production in this case can be formulated as follows:

(SI) Equations of non-trivial balance in "prices":

$$xC_2 = y\left(V_1 + M_{1V}\right) \tag{107}$$

$$xC_3 = zM_{1m} \tag{108}$$

$$y\left(V_3 + M_{3V}\right) = zM_{2m} \tag{109}$$

Second, input and output flows (in "values") in economy with simple reproduction satisfy to trivial balance-conditions:

(SII) Equations for input-output balance (in "values").

$$C_1 + C_2 + C_3 = C = C_1 + V_1\left(1+m\right) \tag{110}$$

$$V + M_V = C_2 + V_2\left(1+m\right) \tag{111}$$

$$M_m = C_3 + V_3(1+m) \qquad (112)$$

Equation (112) follows from equations (110) – (111).

Third, we have condition for "prices of production": prices of balanced exchange coincide with "production prices":

(SIII) Condition for "prices of production":

$$yM_{nV} = \alpha_n r(C_n x + V_n y) \qquad n = 1;2;3 \qquad (113)$$

$$zM_{nm} = (1-\alpha_n) r(C_n x + V_n y) \qquad (114)$$

Finally, we postulate Marx's transformation rules:

(SIV) Transformation rules:

$$M = mV = M_V + M_m = yM_V + zM_m \qquad (115)$$

$$C + V = Cx + Vy \qquad (116)$$

Parameters of this model are connected by relations:

$$M_V = M_{1V} + M_{2V} + M_{3V} \qquad (117)$$

$$M_m = M_{1m} + M_{2m} + M_{3m} \qquad (118)$$

So, the statement problem in this case includes four systems of equations (SI) – (SIV). The solution depends on six arbitrary positive values: $C_1;C_2;C_3;V_1;V_2;\alpha_2$. Unknown variables of our system of equations (SI) – (SIV) are some functions of these parameters.

Solution is "realistic" if all parameters and variables are positive and economically reasonable.

SOLUTION OF THE SYSTEM (SI) – (SIV).

Equations (108), (109), (113), (114) give the following relations:

$$t = \frac{x}{y} = \frac{V_1}{C_2 - \beta_1 C_3}; \quad \beta_1 = \frac{\alpha_1}{1-\alpha_1}; \quad x = ty. \qquad (119)$$

$$r = \frac{C_3 t}{(1-\alpha_1)(C_1 t + V_1)} = \frac{C_3}{(1-\alpha_1)(C_1 + C_2 - \beta_1 C_3)} = \frac{r_0}{(1-\alpha_1)(1-\beta_1 r_0)}; \qquad r_0 = \frac{C_3}{C_1 + C_2}. \quad (120)$$

Formula (120) coincide with formula (31) if $\alpha_1 = 0$.

Formulas (109) and (114) give the next relations:

$$\frac{y}{z} = \frac{M_{2m}}{V_3 + M_{3V}} = \frac{M_{3m}}{r(1-\alpha_3)(C_3 t + V_3)} \qquad (121)$$

Consistency condition for the system (SI):

$$\frac{M_{1m}}{M_{2m}} = \frac{C_3(V_1 + M_{1V})}{C_2(V_3 + M_{3V})} \qquad (122)$$

Equations (110) and (111) lead to the following identities:

$$m = \frac{C_2 + C_3 - V_1}{V_1} \qquad (123)$$

$$M_V = C_2 + V_2(1+m) - V \qquad (124)$$

$$M = mV \qquad (125)$$

$$M_m = M - M_V \qquad (126)$$

Relations (119) – (126) are follow from systems (SI) – (SIII).

The following relations follow from transformational rules (SIV):

$$y = \frac{C+V}{Ct+V} \quad (127)$$

$$z = \frac{M - yM_V}{M_m} = \frac{C_3 ty}{M_{1m}} \quad (128)$$

We have also the next relations (follow from (108), (113) and (114)):

$$M_{1m} = \frac{C_3 ty}{z} = \frac{(1-\alpha_1) r (C_1 x + V_1 y)}{z} \quad (129)$$

$$M_{1V} = \alpha_1 r (C_1 t + V_1) \quad (130)$$

Parameters $\alpha_1; \alpha_2; \alpha_3$ satisfy the following conditions which can be deduced after some computations.

1) Parameter α_1 is smallest root of the following square equation:

$$\left[P(C_2 + C_3) + V_1 C_1 C_3\right]\alpha_1^2 + \left[V_1 C_3 (C_2 - C_1) - PC_2 - Q(C_2 + C_3)\right]\alpha_1 + (QC_2 - V_1 C_2 C_3) = 0 \quad (131)$$

$$P = V_1 C_3 + C(M_V + V_3) \quad (132)$$

$$Q = (M_V + V_3)(C_1 + C_2) - V_2 C_3 \quad (133)$$

2) Parameter α_2 is arbitrary number in interval (0;1).

$0 < \alpha_2 < 1$ - is arbitrary number. (134)

We can choose this parameter so that to obtain the realistic solution.

3) Parameter α_3 satisfies to relation:

$$\alpha_3 = (1-\alpha_2) \cdot \left(\frac{C_2 x + V_2 y}{C_3 x + V_3 y}\right) - \frac{yV_3}{r(C_3 x + V_3 y)} \quad (135)$$

Value V can be found if organic composition k of the capital is known:

$$V = kC = \frac{rC}{m-r} \quad (136)$$

The following square equation for parameter k can be obtained after very long computations (it is necessary to substitute (136) and (124) into (131)):

$$Ak^2 + Bk + E = 0 \quad (137)$$

$$A = A_1 X^2 + mC(XB_1 + mCD_1) \quad (138)$$

$$B = -(2C_3 X A_1 + mCC_3 B_1) \quad (139)$$

$$E = A_1 C_3^2 \quad (140)$$

$$A_1 = V_1 C_1 C_3 + (C_2 + C_3)\left[C(C_2 + mV_2) - V_1(C_1 + C_2)\right] \quad (141)$$

$$\begin{aligned} B_1 = {} & V_1 C_3 (C_2 - C_1) - C_2\left[C(C_2 + mV_2) - V_1(C_1 + C_2)\right] - \\ & -(C_2 + C_3)\left[V_2(C_1 + C_2)(1+m) + (C_2 - V_1 - V_2)(C_1 + C_2) - V_2 C_3\right] \end{aligned} \quad (142)$$

$$D_1 = C_2\left[V_2(C_1 + C_2)(1+m) + (C_2 - V_1 - V_2)(C_1 + C_2) - V_2 C_3\right] - V_1 C_2 C_3 \quad (143)$$

$$X = m(C_1 + C_2) - C_3 \quad (144)$$

Value V_3 can be calculated as follows:

$$V_3 = V - V_1 - V_2 \quad (145)$$

SOLUTION IS COMPLETED.

Supplement 3 contains numerical example of transformation in the model-2[7].

X. Conclusion.

This (first) part of paper was devoted to the dissolution of transformation problem in the Model-1 and Model-2 of simple production. Realistic solutions exist both in Model-1 and in Model-2 if “non-trivial balance conditions” are carried out in economy. Model-1 and Model-2 strongly simplify reality. We will consider more realistic Model-4 in the following second part of this paper. Model-4 is more realistic model in which “labor of capitalists” (as entrepreneurs) and tax-system are taken into account. Model-1 and Model-2 (with “non-trivial balance-conditions”) describe early capitalist economy without tax-system and without "labor of capitalists (as entrepreneurs)". Marx postulated "non-trivial conditions of balanced exchange" for the early capitalist economy in the second volume of “Capital” (chapter XX). We have discussed shortly why Marx’s non-trivial conditions most likely were carried out with high precision in early capitalist economy (during pre-industrial epoch). Merchant-capitalism prevailed. Merchants invested money-capital in different kinds of production. Therefore the capital of each department consisted of many separate capitals of different merchants. “The law of big numbers” was equalizing some critical parameters of economy in frame of more realistic Model-4 that ensured the execution of Marx’s non-trivial conditions.

“Transformation problem” came about from Bortkiewicz’ (1907) paper in which he stated the question whether Marx’s transformation rules can be executed in Model-1. He answered - “no”. Our analysis demonstrates that it is not so. Solution exists always in the model-1 if Marx’s “non-trivial conditions of balanced exchange” are taken into account. Solution exists also in the Model-2. Bortkiewicz has not taken into account non-trivial balance-conditions existed in early capitalist economy with simple reproduction. As consequence he obtained more wide set of solutions. Just and only solutions which don’t satisfy to Marx’s “non-trivial conditions of the balance” don’t satisfy also to Marx’s rules of transformation.

Transformation problem in the model-4 will be considered in the second part of this paper. Solution in frame of this more realistic model exists even without imposing of special “non-trivial conditions of balanced exchange”.

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[7] Transformation problem in Model-2 was considered recently in paper Calleja (2010). System of equations (RSD) on page 43 of this paper describes balance-conditions in “values”. System of equations (RSP) on page 44 describes 1) balance-conditions in “prices of production” and 2) Marx’s “transformational rules”. It gives 12 equations for 12 variables. Author illustrates solution by means of numerical examples. Emilio Calleja’s equations are fulfilled in our solution but Calleja’s solutions don’t satisfy very important condition:

$$P_{21} = \left(C_{21}\pi^{C} + V_{21}\pi^{V}\right)\left(1+r\right) = D_{21}\pi^{V}\ {}^{7}$$

Let’s compare for example Tables 13.1 (value-structure in form “labor costs”) and 13.3 (for “prices of production”). We see that $D_{21} = 300 = P_{21}$ though $\pi^{V} = 0.806 < 1$. Price of production of goods of sub-department IIa is less than value. Consequently volume of output after transformation obviously became larger. Consequently Emilio Calleja supposes that transformation is accompanied by some change in volumes of production. This result is incompatible with problem statement of current transformation when we describe algorithm by means of which total surplus-value perpetually redistributed between capitalists whereas volumes of current production are fixed.

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SUPPLEMENT I. Numerical examples of value-structure.

Example 1. Value-structure (SV) of exchange based on "values".

	C	V	M	m	W
I	280.00	420.00	300.00	0.71	1000.00
II	420.00	105.00	75.00	0.71	600.00
III	300.00	75.00	53.57	0.71	428.57
Σ:	1000.00	600.00	**428.57**	0.71	**2028.57**

The following parameters were used: $a = 0.4$; $b = 0.3$; $k = 0.6$.

Example 2. Value-structure of exchange in which "values" and "prices of production" coincide.

Value of products consumed in each department (labor commanded):					
	C (means of production)	V (consumer goods)	M (luxury goods)	m	W
I	420.00	280.00	300.00	1.071	1000.00
II	280.00	186.67	200.00	1.071	666.67
III	300.00	200.00	214.29	1.071	714.29
SUM:	1000.00	666.67	**714.29**	1.071	**2380.95**
Value of product produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	280.00	300.00	1.071	1000.00
II	280.00	186.67	200.00	1.071	666.67
III	300.00	200.00	214.29	1.071	**714.29**
SUM:	1000.00	666.67	**714.29**	1.071	**2380.95**
Prices of production:					
	C (means of production)	V (consumer goods)	M (luxury goods)	r	W
I	420.00	280.00	300.00	0.429	1000.00
II	280.00	186.67	200.00	0.429	666.67
III	300.00	200.00	214.29	0.429	**714.29**
SUM:	1000.00	666.67	**714.29**	0.429	**2380.95**

The same color corresponds to the equal quantities.

The following parameters were used: $a = 0.6$; $b = 0.3$. Calculations give the following values: $k = 0.667$; $r = 0.429$.

Example 3. Value-structure (SP) of exchange based on "prices of production".

Value of products consumed in each department (labor commanded):					
	C (means of production)	V (consumer goods)	M (luxury goods)	m	W
I	420.00	338.33	334.62	0.989	1092.95
II	280.00	711.67	415.38	0.584	1407.05
III	300.00	450.00	321.43	0.714	1071.43
SUM:	1000.00	1500.00	**1071.43**	0.714	**3571.43**
Value of product produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	338.33	241.67	0.714	1000.00
II	280.00	711.67	508.33	0.714	1500.00
III	300.00	450.00	321.43	0.714	**1071.43**
SUM:	1000.00	1500.00	**1071.43**	0.714	**3571.43**
Prices of production:					
	C (means of production)	V (consumer goods)	M (luxury goods)	r	W
I	468.46	312.31	334.62	0.429	1115.38
II	312.31	656.92	415.38	0.429	1384.62
III	334.62	415.38	321.43	0.429	**1071.43**
SUM:	1115.38	1384.62	**1071.43**	0.429	**3571.43**

The following parameters were used: $a = 0.6$, $b = 0.3$, $k = 1.5$.
Calculations give: $x = 1.1154$, $y = 0.9231$, $z = 1$, $r = 0.429$.

SUPPLEMENT II. Numerical examples of modelling of historical transformation.

Let's consider numerical examples which illustrate how the exchange based on "values" could be transformed into the exchange based on "prices of production". Consider simple case when only third department is modernized. Let technological coefficient γ_3 and rate of surplus-value be varying whereas all other values are constant.

Exchange based on "values" is initial state of economy (before transformation). This initial state in our model can be given by means of the following relations:

(SII.1) $m = m_1 = m_2 = m_3$

(SII.2) $\gamma_2 = \gamma_3$

(SII.3) $x = y = z = 1$

Example 1. Parameter γ_3 increases during transition period.

Let's take the following initial values of parameters: $\gamma_1 = 0.3 > \gamma_2$; $\gamma_2 = \gamma_3 = 0.1$; $m = 1$.

The following quantities can be calculated:

(SII.4) $\beta = \frac{1}{1+m}$

(SII.5) $k = \frac{V}{C} = \frac{1-\gamma_1}{\gamma_2 + m\gamma_3}$

(SII.6) $t = \frac{V_1}{C_2}$

(SII.7) $y = \frac{M_2}{V_3}$

(SII.8) $x = ty$

(SII.9) $\frac{m_1}{m_2} = \frac{k_2}{k_3} = \frac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)}$

(SII.10) $r_1 = \frac{m\gamma_3(1-\gamma_1)}{\gamma_2 + m\gamma_1\gamma_3}$

(SII.11) $r_2 = r_3 = \frac{m(1-\gamma_3)}{1+m\gamma_3}$

Table I(SII). Initial Matrix of Exchange based on "values".

Value of products consumed in each department (labor commanded):							
	C	V	M	m	SUM:	r	k
I	300	350	350	1.00	1000	0.54	1.17
II	350	1575	1575	1.00	3500	0.82	4.50
III	350	1575	1575	1.00	3500	0.82	4.50
SUM:	1000	3500	3500	1.00	8000	0.78	3.50

Transformation of values into production prices in our model occurs on account of modernization of third department. Modernization is possible if this process doesn't decrease the rate of profit in third department. Let's consider "threshold case" when rate of profit in third department is constant during transition period. We will change parameters stepwise. Transition period is modeled as recurrent process consisting of many sub-periods with different values of parameters. Let's designate parameters in each sub-period as $m^{(n)};\gamma_3^{(n)}$ where n is the number of sub-period. The rate of profit is the same in each sub-period n if rate of surplus-value satisfy to the following relation:

(SII.12) $m^{(n)} = \dfrac{r_3}{1-\gamma_3^{(n)}(1+r_3)}$

Recurrent relations are formulated as follows:

(SII.13) $\gamma_3^{(n+1)} = \gamma_3^{(n)} + h\cdot\left(r_3^{(n)} - r_1^{(n)}\right)$

(SII.14) $m_3^{(n)} = m^{(n)}$

We took value $h = 0.8$. Procedure of transformation based on recurrent relations (SII.12-14) leads to the exchange based on "production prices".

Table II(SII). Exchange based on production prices.

Value of products consumed in each department (labor commanded):							
	C	V	M	m	SUM:	r	k
I	300.00	326.30	539.99	1.65	1166.30	0.86	1.09
II	250.10	1049.24	1035.35	0.99	2334.70	0.80	4.20
III	449.90	1125.45	1288.92	1.15	2864.27	0.82	2.50
SUM:	1000.00	2500.99	**2864.27**	1.15	**6365.26**	0.82	2.50

Matrix of input-flows in "production prices".							
	C	V	M	m	SUM:	r	k
I	360.07	300.18	539.99	1.799	**1200.25**	0.818	0.83
II	300.18	965.25	1035.35	1.073	**2300.79**	0.818	3.22
III	539.99	1035.35	1288.92	1.245	**2864.27**	0.818	1.92
SUM:	**1200.25**	**2300.79**	**2864.27**	1.245	**6365.30**	0.818	1.92
Value of product produced in departments (labor cost):							
	C	V	M	m	SUM:	r	k
I	300.00	326.30	373.70	1.15	1000.00	0.60	1.09
II	250.10	1049.24	1201.65	1.15	2500.99	0.92	4.20
III	449.90	1125.45	1288.92	1.15	**2864.27**	0.82	2.50
SUM:	1000.00	2500.99	**2864.27**	1.15	**6365.26**	0.82	2.50
gamma1	0.300	x	1.200	m1	1.655	m	1.145
gamma2	0.100	y	0.920	m2	0.987	r	0.818
gamma3	0.157	z	1.000	m3	1.145	k	2.501

The same color means the equal quantities.
Graphs bellow illustrate the dynamics of different economic quantities during the transition period.

Figure 1(SII). Rates of surplus-value.

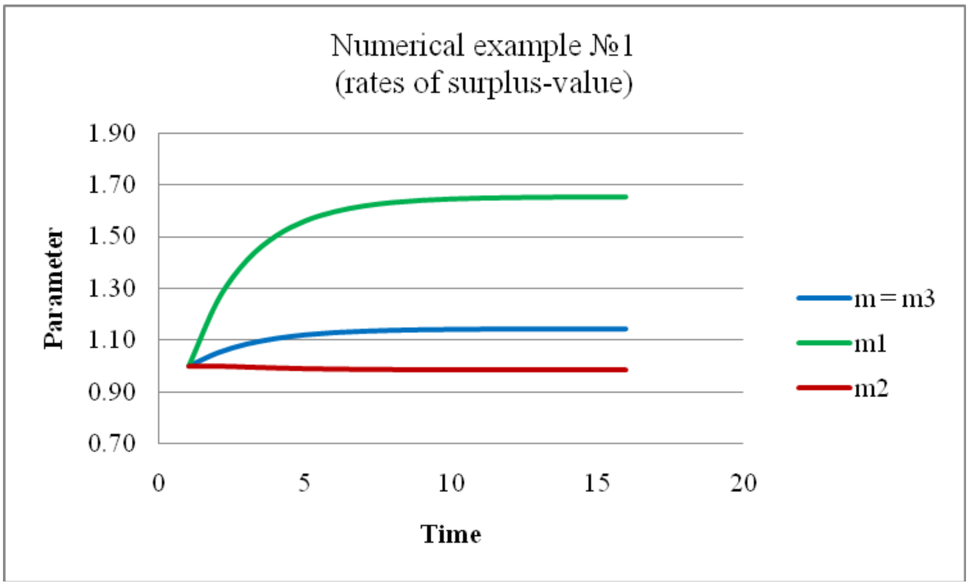


Figure 2(SII). Technological coefficients $\gamma_1; \gamma_2; \gamma_3$.

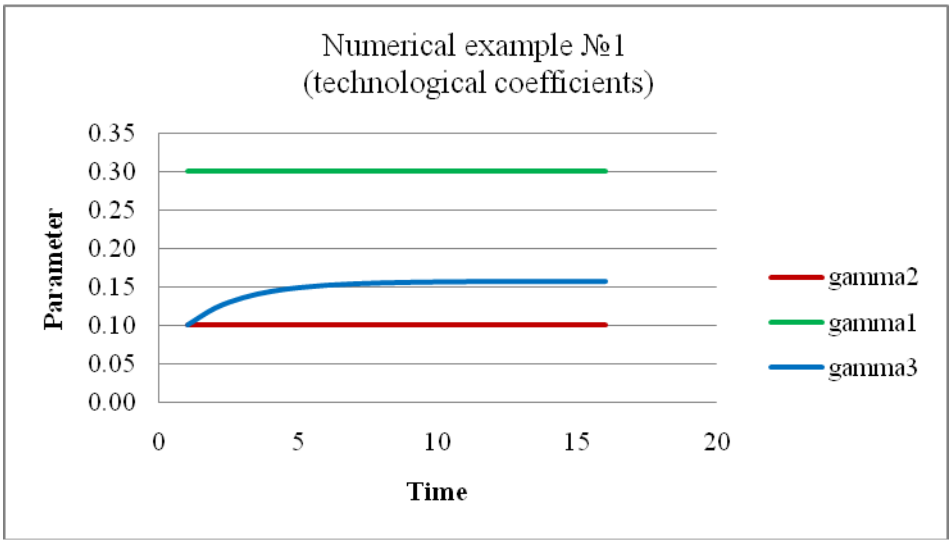


Figure 3(SII). Rates of profit r_1; $r_2 = r_3$.

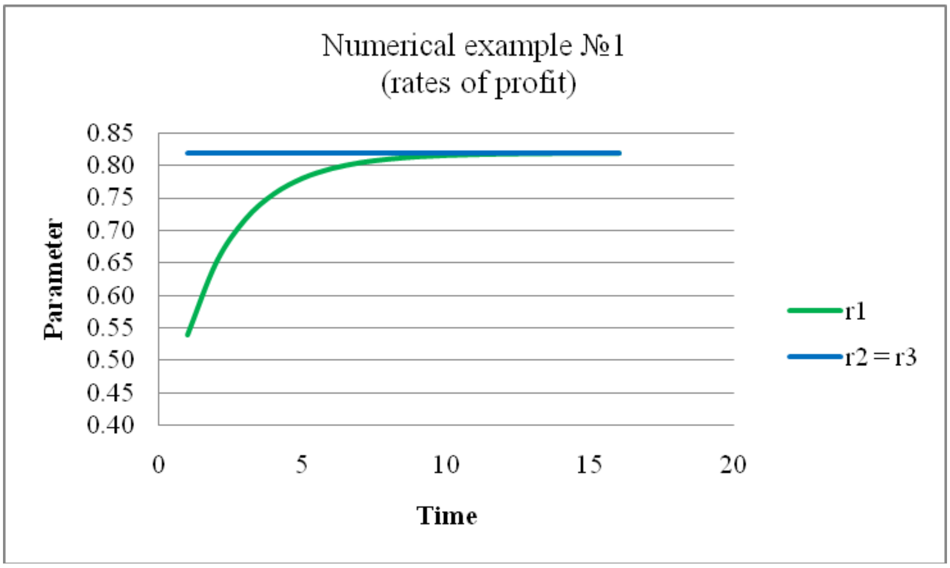


Figure 4(SII). Temporary equilibrium prices $x; y;\ z = 1$.

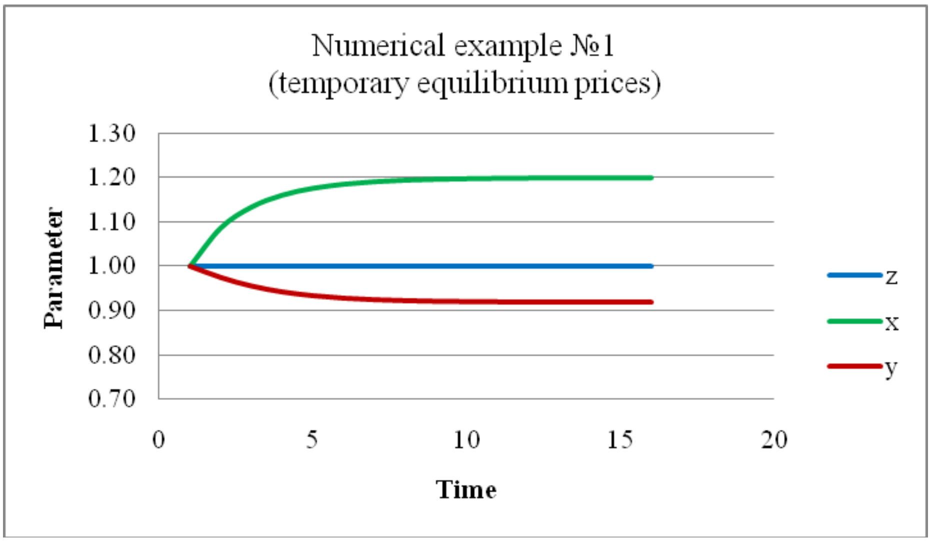


Figure 5(SII). Organic compositions of capital (in values) $k_i = \dfrac{V_i}{C_i};\ i = \text{I;II;III}$.

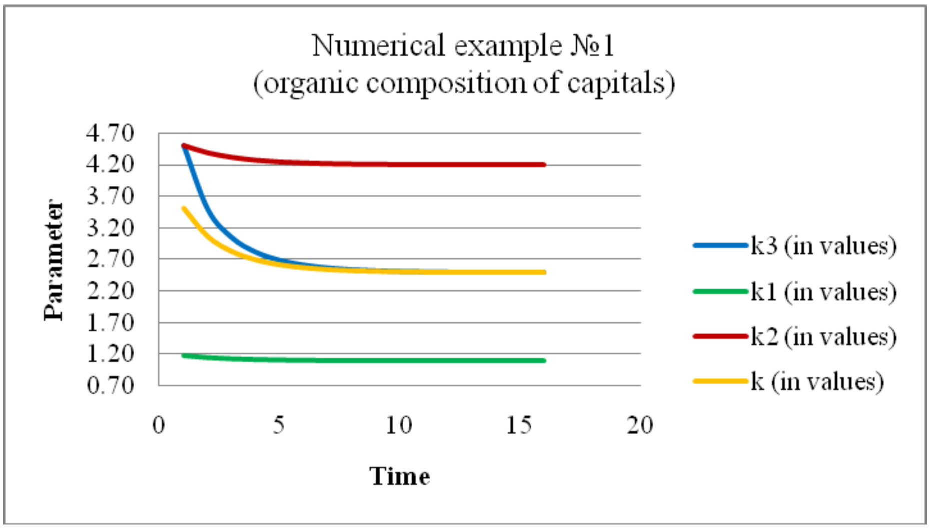
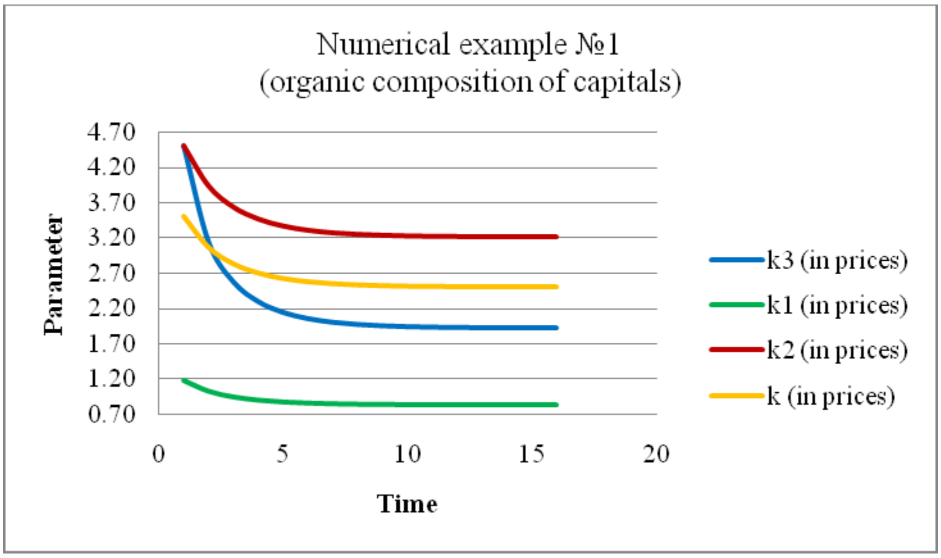
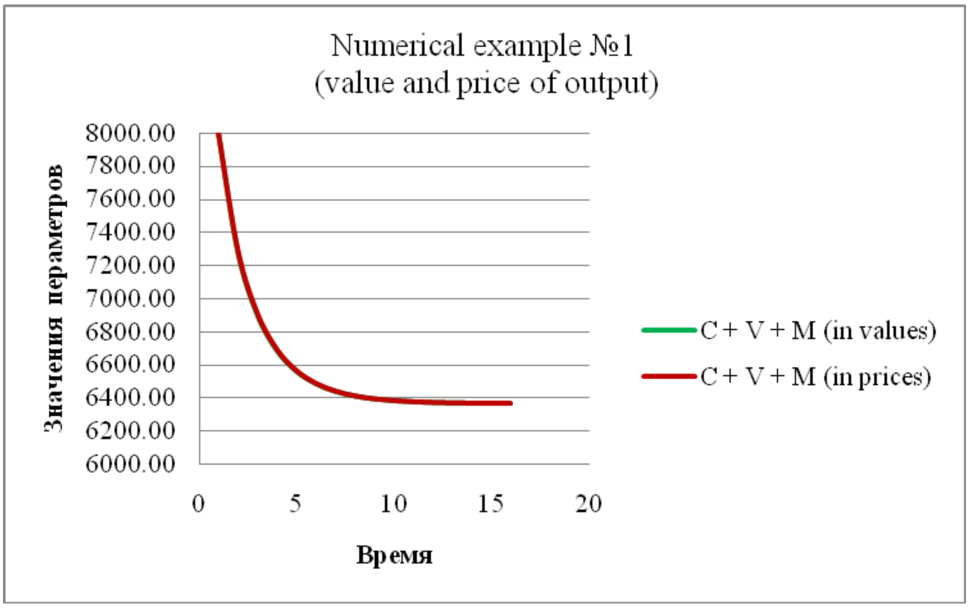


Figure 6(SII). Organic compositions of capital (in temporary equilibrium prices).



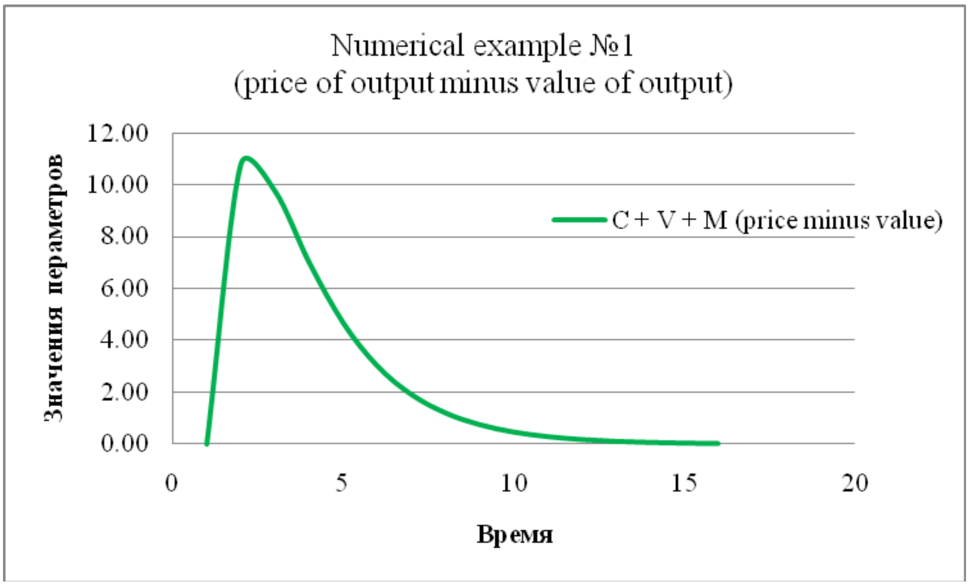
Marx's transformation rules are fulfilled to high precision in each sub-period of transition period. The rule (RI) (total surplus-value is equal to total profit) is fulfilled exactly whereas the rule (RII) (total value of output is equal to total temporary equilibrium price of output) is fulfilled to high precision.

Figure 7(SII). Total value and price of output.



The difference between value and price of output is no more than 0.15% of output value. We see that graphs in Figure 7(SII) almost coincide. Graph of the difference is depictured in Figure 8(SII).

Figure 8(SII). The difference between value and price of output.



Example №2. Parameter γ_3 decreases during transition period.

Let's take the following initial values of parameters $\gamma_1 = 0.3 < \gamma_2$; $\gamma_2 = \gamma_3 = 0.5$; $m = 1$.

Recurrent relations are the same as in the first example. We used value $h = 0.27$ for the second model.

Table III(SII). Initial Matrix of Exchange based on "values".

Value of products consumed in each department (labor commanded):							
	C	V	M	m	Σ:	r	k
I	300	350	350	1.00	1000	0.54	1.17
II	350	175	175	1.00	700	0.33	0.50
III	350	175	175	1.00	700	0.33	0.50
Σ:	1000	700	700	1.00	2400	0.41	0.70

Table IV(SII). Exchange based on production prices (after transformation).

Value of products consumed in each department (labor commanded):							
	C	V	M	m	Σ:	r	k
I	300.00	410.85	239.25	0.58	950.10	0.337	1.37
II	449.96	264.10	235.77	0.89	949.83	0.330	0.59
III	250.04	224.98	158.34	0.70	633.36	0.333	0.90
Σ:	1000.00	899.93	633.36	0.70	2533.29	0.333	0.90

Matrix of input-flows in "production prices".							
	C	V	M	m	Σ:	r	k
I	287.06	430.55	239.25	0.556	956.85	0.333	1.50
II	430.55	276.76	235.77	0.852	943.08	0.333	0.64
III	239.25	235.77	158.34	0.672	633.36	0.333	0.99
Σ:	956.85	943.08	633.36	0.672	2533.29	0.333	0.99
Value of product produced in departments (labor cost):							
	C	V	M	m	Σ:	r	k
I	300.00	410.85	289.15	0.70	1000.00	0.41	1.37
II	449.96	264.10	185.87	0.70	899.93	0.26	0.59
III	250.04	224.98	158.34	0.70	633.36	0.33	0.90
Σ:	1000.00	899.93	633.36	0.70	2533.29	0.33	0.90
gamma1	0.300	x	0.957	m1	0.582	m	0.704
gamma2	0.500	y	1.048	m2	0.893	r	0.333
gamma3	0.395	z	1.000	m3	0.704	k	0.900

The same color means the equal quantities.

Graphs bellow illustrates the dynamics of different economic values during the transition period.

Figure 9(SII). Rates of surplus-value.

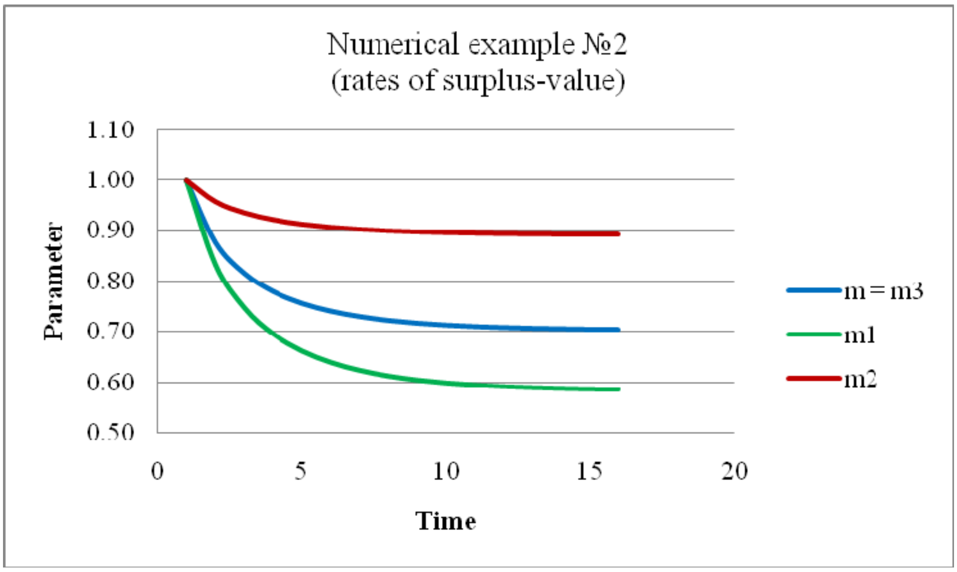


Figure 10(SII). Technological coefficients $\gamma_1;\gamma_2;\gamma_3$.

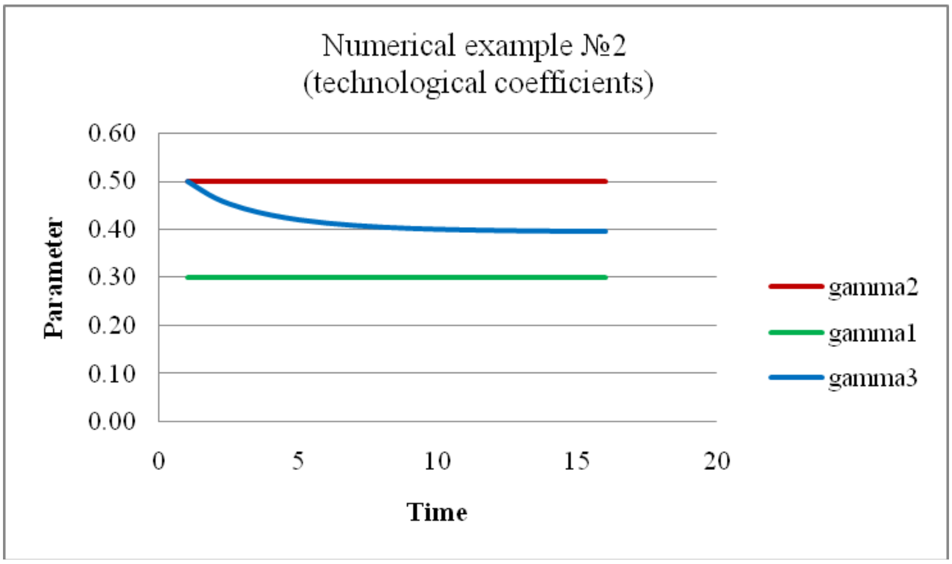


Figure 11(SII). Rates of profit $r_1;\ r_2 = r_3$.

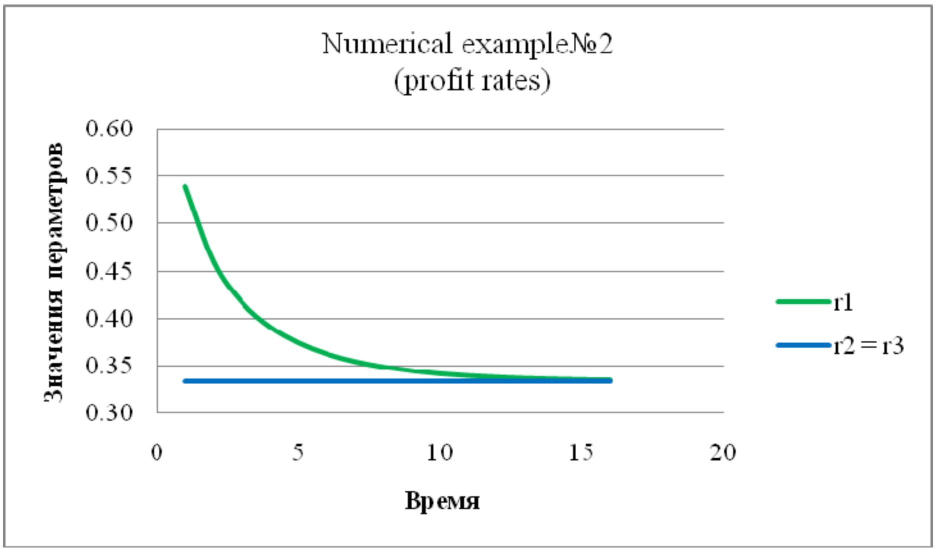


Figure 12(SII). Temporary equilibrium prices $x; y; z = 1$.

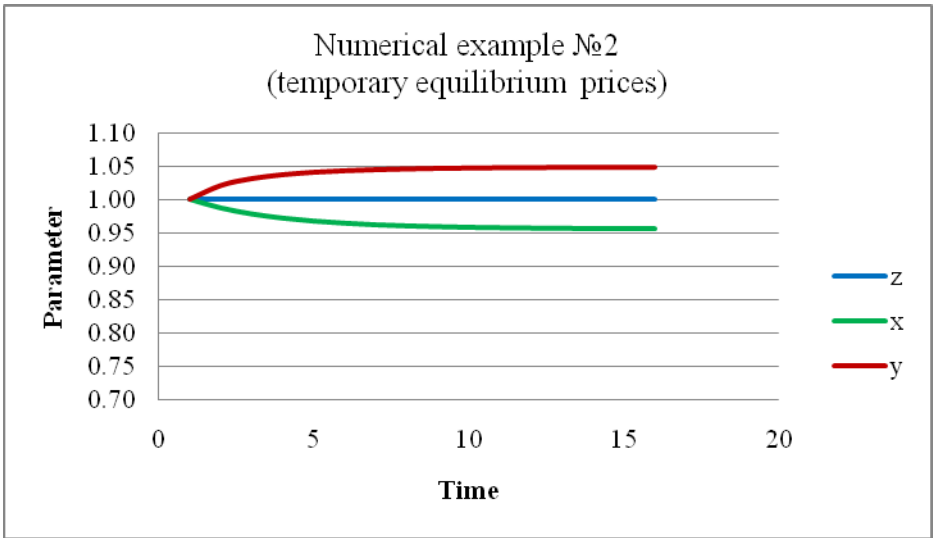


Figure 13(SII). Organic compositions of capital (in values) $k_i = \dfrac{V_i}{C_i}$; $i = \text{I;II;III}$.

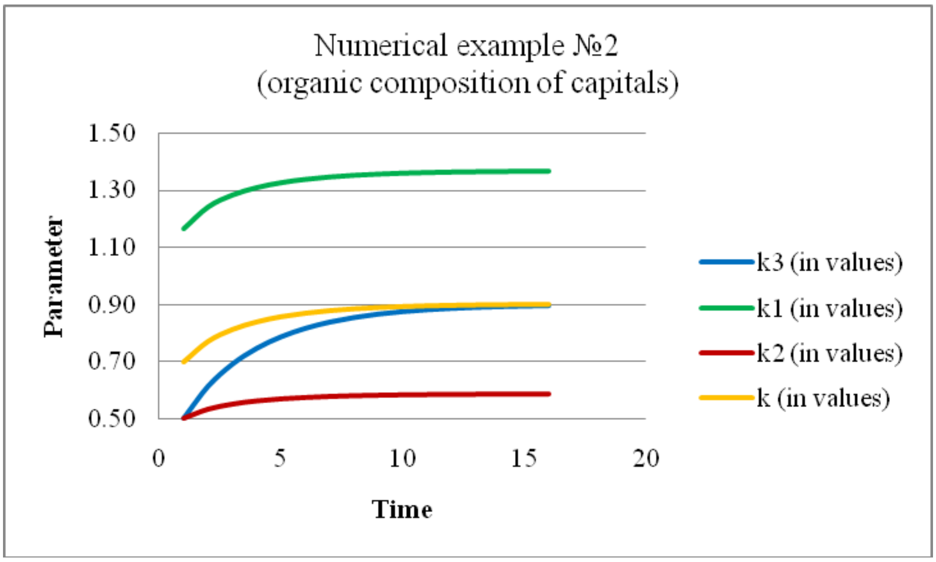


Figure 14(SII). Organic compositions of capital (in temporary equilibrium prices).

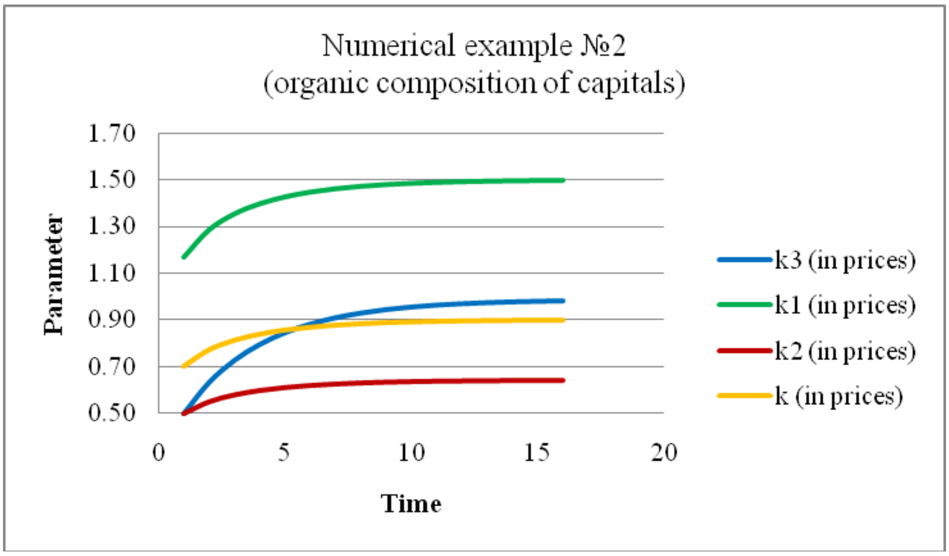


Figure 15(SII). Total value and price of output.

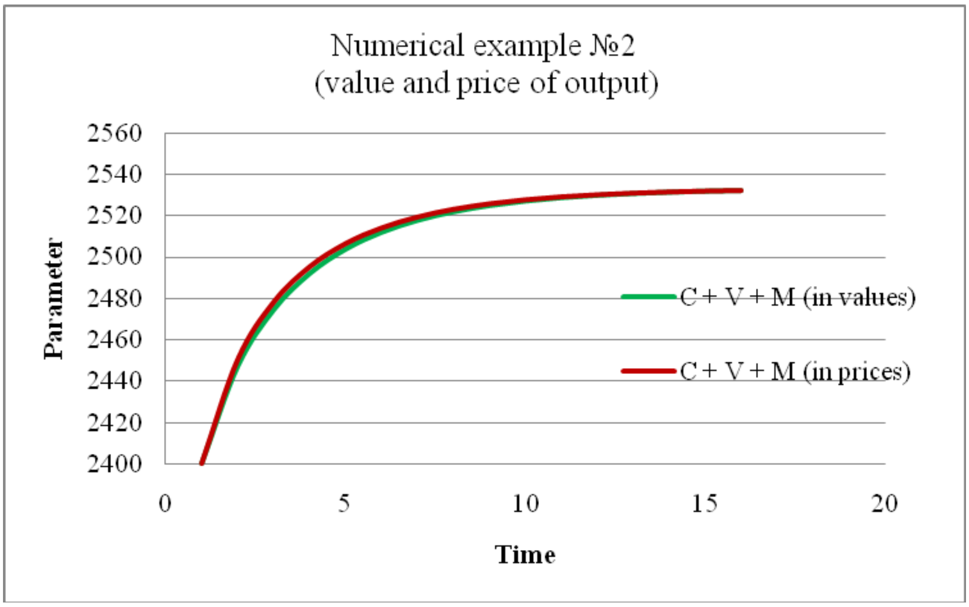
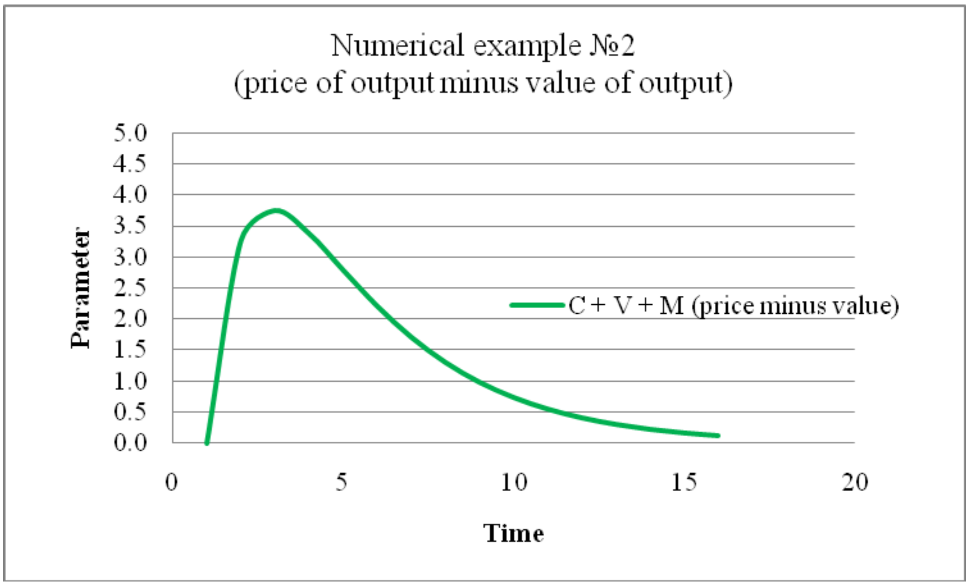


Figure 16(SII). The difference between value and price of output.



Short discussion of results.

The qualitative properties of these models are presented in the following Table.

Table V. Qualitative properties of models (numerical examples №1 and №2).

	gamma3	m = m3	m1	m2	r1	k1	k2	k3	k	x	y	C+V+M
I	↑	↑	↑	↓	↑	↓	↓	↓	↓	↑	↓	↓
II	↓	↓	↓	↓	↓	↑	↑	↑	↑	↓	↑	↑

Symbol ↑ (↓) designate the increase (the decrease) of quantity during transition period. We see that properties of these solutions are antithetical. Transformation rules are fulfilled to high precision during transition period. Output (in values and in prices) decreases in the first case and it grows in the second case. Possibly jump in rate of growth during so-called "industrial revolution" can be explained partially by the influence of transformation.

SUPPLEMENT III. Numerical example of solution in the Model-2 (for asymmetric matrix of input-flows).

Value of products consumed in each department (labor commanded):						
	C (means of production)	V (consumer goods)	Mv (consumer goods)	Mm (luxury goods)	m	W
I	420.00	340.00	105.80	220.74	0.960	865.80
II	370.00	940.00	235.09	307.65	0.577	1545.09
III	210.00	225.17	127.47	65.73	0.858	562.64
SUM:	**1000.00**	1505.17	468.36	**594.12**	0.706	**3567.65**
	V + Mv =	**1973.53**	M = Mv + Mm =	**1062.47**		
CAPITAL =		**2505.17**		r = M' : (C' + V') =		**0.4241**

Value of products produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	340.00	240.00	0.706	**1000.00**
II	370.00	940.00	663.53	0.706	**1973.53**
III	210.00	225.17	158.95	0.706	**594.12**
SUM:	1000.00	1505.17	**1062.47**	0.706	**3567.65**
CAPITAL =		**2505.17**		r = M' : (C' + V') =	**0.4241**

Prices of production:						
	C (means of production)	V (consumer goods)	Mv (consumer goods)	Mm (luxury goods)	r	W
I	467.79	314.30	97.80	233.89	0.424	**1113.78**
II	412.10	868.94	217.32	325.98	0.424	**1824.35**
III	233.89	208.15	117.83	69.64	0.424	**629.52**
SUM:	**1113.78**	1391.39	432.95	**629.52**	0.424	**3567.65**
CAPITAL =		**2505.17**		r = P : (C + V) =		**0.4241**
	V + Mv =	**1824.35**	P = Mv + Mm =	**1062.47**		
	x	**1.114**	**alpha1**	**0.295**	**k**	**1.505**
	y	**0.924**	**alpha2**	**0.400**		
	z	**1.060**	**alpha3**	**0.629**		
	m	**0.706**	**r**	**0.424**		

SUPPLEMENT IV.

Technical Excel-file contain programs of calculations.
Worksheet “SV” = (SV)-structure;
Worksheet “SP” = (SP)-structure;
Worksheet “SP-SV” = structure when exchange based on “values” coincides with exchange based on “production prices”;
Worksheet “Ex1” = model of “historical transformation” (example №1);
Worksheet “Ex2” = model of “historical transformation” (example №2);
Worksheet “Mod-2” = numerical example of solution in the Model-2.
Worksheet “Mod-2 (Marx)” = Marx’s numerical example of solution in the Model-2.